THE FOUNDATIONS OF PROBABILITY

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1. Introduction. Probability is a branch of mathematics. It is not a branch of experimental science nor of armchair philosophy, it is neither physics nor logic. This is not to say that the experimenter and the philosopher should not discuss probability from their points of view. They should, and they do. The situation is analogous to that in geometry. No one denies that the physicist and the philosopher have made valuable contributions to our understanding of the space concept, nor, in spite of this, that geometry is a rigorous part of modern mathematics.

Like Euclidean geometry, and for that matter like most mathematical theories, probability has four aspects: axiomatization, development, coordinatization, and application. We proceed to explain our use of these words.

"Axiomatization" is clear. We all know that the study of geometry begins with a list of undefined terms and a list of postulates. It is important in this connection to remember two facts. First: the selection of the list of terms and postulates is not entirely arbitrary, but is derived only after a thorough examination of our intuitive notions of the subject. Second: the selection of terms and postulates is not uniquely determined. When several different axiomatizations of the same subject exist then only extra mathematical considerations, such as practical convenience or personal prejudice, can lead us to prefer one among the many. The greater part of this paper is devoted to a prepostulational examination of probability. The axiomatic system to which this examination leads is not the only possible approach to probability, but it is the approach which has been adopted by the majority of workers in this field.

By "development" we mean simply the main part of the theory, the definitions and theorems which chiefly occupy the professional mathematician. "Coordinatization" is a general process the most familiar instance of which is the proof of the equivalence of the synthetic and analytic aspects of Euclidean geometry. The isomorphism of a finite group to a group of permutations and the representation of an algebra by matrices are further examples of this process. Properly speaking coordinatization is just one of the theorems belonging to development, but a theorem of such fundamental implications that it effects basic changes in the appearance, methods, and results of the entire theory.

The hardest philosophical problem in geometry as well as in probability is the problem of "application." Do the theorems derived from the postulates reflect any light on the physical world which suggested them, and if so, how and why?

The purpose of this paper is exposition, exposition intended to convince the professional mathematician that probability is mathematics. To this end we shall discuss the four features just enumerated. The paper contains almost no proofs, very few precise definitions and theorems, and many heuristic derivations. Despite however the small number of rigorous statements, they form the

foundation on which the remainder is built. For the convenience of the reader they are italicized. If these italicized statements are lifted from their context and read consecutively, they will furnish at least a partial answer to the question "what is probability?"

2. Boolean algebra. The principal undefined term in probability theory is "event." Intuitively speaking an event is one of the possible outcomes of some physical experiment.

To take a rather popular example consider the experiment of rolling an ordinary six-sided die and observing the number v($=1, 2, 3, 4, 5,$ or 6) showing on the top face of the die. "The number v is even"—"it is less than 4"—"it is equal to 6"—each such statement corresponds to a possible outcome of the experiment. From this point of view there are as many events associated with the experiment as there are combinations of the first six positive integers taken any number at a time. If for the sake of aesthetic completeness and later convenience we consider also the impossible event, "the number v is not equal to any of the first six positive integers," then there are altogether 2^6 admissible events associated with the experiment of the rolling die. For the purpose of studying this example in more detail let us introduce some notation. We write $\{246\}$ for the event "v is even," $\{123\}$ for "v is less than 4," and so on. The impossible event and the certain event ($=\{123456\}$) deserve special names: we reserve for them the symbols o and e respectively.

Everyday language concerning events uses such phrases as these: "two events a and b are incompatible or mutually exclusive," "the event a is the opposite of the event b or complementary to b," "the event a consists of the simultaneous occurrence of b and c," "the event a consists of the occurrence of at least one of the two events b and c." Such phrases suggest that there are relations between events and ways of making new events out of old that should certainly be a part of their mathematical theory.

The notion of complementary event is probably closest to the surface. If a is an event we denote the complementary event by a': an experiment one of whose outcomes is a will be said to result in a' if and only if it does not result in a. Thus if $a=\{246\}$ then $a'=\{135\}$. We may also introduce combinations of events suggested by the logical concepts of "and" and "or." With any two events a and b we associate their "join" $a\cup b$ (also called union or sum and often denoted by $a+b$), and their "meet" $a\cap b$ (or intersection or product, often denoted by ab). Here $a\cup b$ occurs if and only if at least one of the two events a or b occurs, while $a\cap b$ occurs if and only if both a and b occur. Thus if $a=\{246\}$ and $b=\{123\}$ then $a\cup b=\{12346\}$ and $a\cap b=\{2\}$.

The operations a', $a\cup b$, and $a\cap b$ satisfy some simple algebraic laws. It is clear for example that both the expressions $a\cup b$ and $a\cap b$ are independent of the order of the terms (commutative law), and that neither of the expressions $a\cup b\cup c$ and $a\cap b\cap c$ depends on the order in which the two indicated operations are performed (associative law). These facts are intuitively obvious from the verbal definition of the operations and are easily verified in any finite case such

as the rolling die. There are many other similar identities satisfied by these methods of combining events: the following is a list of the most important ones.

$$o' = e \qquad (a')' = a \qquad e' = o$$

$$(a \cap b)' = a' \cup b' \qquad (a \cup b)' = a' \cap b'$$

$$a \cap a' = o \qquad a \cup a' = e$$

$$o \cap a = o \qquad o \cup a = a$$

$$e \cap a = a \qquad e \cup a = e$$

$$a \cap b = b \cap a \qquad a \cup b = b \cup a$$

$$(a \cap b) \cap c = a \cap (b \cap c) \qquad (a \cup b) \cup c = a \cup (b \cup c)$$

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \qquad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

A system B of elements $o, a, b, \cdots, e$ in which operations a', $a \cup b$, and $a \cap b$ are defined in such a way that each of the above list of identities is satisfied is called a "Boolean algebra." For the traditional theory of probability, concerned with simple gambling games such as the rolling die, in which the total number of possible events is finite, the above heuristic reduction of events to elements of a Boolean algebra is adequate. For situations arising in modern theory and practice, and even for the more complicated gambling games, it is necessary to make an additional assumption. This assumption, in descriptive terms, is that the operations $\cup$ and $\cap$, assumed defined for two elements and immediately extended by mathematical induction to any finite number, should make sense also for an infinite sequence. In other words it is desirable to have an interpretation for symbols such as $a_1 \cup a_2 \cup \cdots$ and $a_1 \cap a_2 \cap \cdots$. In order to phrase precisely this assumption of infinite operations it is necessary to use a few simple facts from the theory of Boolean algebras.

If a and b are any two elements of the Boolean algebra B which satisfy the relation $a \cup b = b$ (or the equivalent relation $a \cap b = a$) we shall write $a \subset b$ and say that "a is smaller than b" or "a is contained in b" or "a implies b." The intuitive interpretation of this relation is as follows: the event a implies the event b, or is contained in the event b, if the occurrence of a is a sub-case of the occurrence of b. Thus in the example of the die $\{123\} \subset \{1234\}$ and "$v = 2$" $\subset$ "v is even." The technical significance of the relation $\subset$ is that the operations $\cup$ and $\cap$ may be defined in terms of it. For example $a \cup b$ is the smallest of all elements which contain both a and b. In more detail: given a and b, consider all c's for which both $a \subset c$ and $b \subset c$. The assertion concerning $a \cup b$ is two fold: first, $a \cup b$ is an admissible c, and second, for any admissible c we have $a \cup b \subset c$. As an example consider $a = \{12\}$ and $b = \{24\}$. The elements $\{1234\}$, $\{1246\}$, $\{12456\}$, $\cdots$ all have the property of containing both a and b. However the element $\{124\}$, which also has that property, is smaller than any other such element, and it is in fact true that $\{12\} \cup \{24\} = \{124\}$.

Motivated by the relation between $\cup$ and $\subset$ we now proceed as follows. Let B be a Boolean algebra. If for every infinite sequence $a_1, a_2, \cdots$ of elements of

B there exists among the elements containing all the a_n a smallest one, say a, we say that B is a σ-algebra and we write $a = a_1 \cup a_2 \cup a_3 \cup \cdots$. Not every Boolean algebra is a σ-algebra; the assumption that B is one (the hypothesis of countable additivity) is an essential restriction.

Perhaps an example, though a somewhat artificial one, might illustrate the need for the added assumption. Suppose that a player determines to roll a die repeatedly until the first time that the number showing on top is 6. Let a_n be the event that the first 6 appears only on the nth roll. The event $a = a_1 \cup a_2 \cup a_3 \cup \cdots$ occurs if and only if the game ends in a finite number of rolls. The occurrence of the opposite event a' is at least logically (even if not practically) conceivable and it seems reasonable to want to include a discussion of it in a general theory of probability. Numerous examples of this kind together with some rather deep lying technical reasons justify therefore the following statement.

The mathematical theory of probability consists of the study of Boolean σ-algebras.

This is not to say that all Boolean σ-algebras are within the domain of probability theory. In general statements concerning such algebras and the relations between their elements are merely qualitative: probability theory differs from the general theory in that it studies also the quantitative aspects of Boolean algebras. In the next section we shall describe and motivate the introduction of numerical probabilities.

3. Measure algebra. When we ask "what is the probability of a certain event?" we expect the answer to be a number, a number associated with the event. In other words probability is a numerically valued function P of events a, that is of elements of a Boolean σ-algebra B, $P = P(a)$. On intuitive and practical grounds we demand that the number $P(a)$ should give information about the occurrence habits of the event a. If in a large number of repetitions of the experiment which may result in the event a we observe that a actually occurs only a quarter of the time (the remaining three quarters of the experiments resulting therefore in a') we may attempt to summarize this fact by saying that $P(a) = 1/4$. Even this very rough first approximation to what is desired yields some suggestive clues concerning the nature of the function P.

If, to begin with, $P(a)$ is to represent the proportion of times that a is expected to occur, then $P(a)$ must be a positive real number, in fact a number in the unit interval $0 \leqq P(a) \leqq 1$. The extreme value 0 has a special significance. Since the impossible event o will never occur, it is clear that we must write $P(o) = 0$. Conversely however if an event a refuses ever to occur, we are tempted to declare its occurrence impossible and thus from the relation $P(a) = 0$ to deduce $a = o$. The other extreme value of $P(a)$ has of course a similar interpretation: $P(a) = 1$ if and only if $a = e$.

The relation between proportion and probability has further consequences. Suppose that a and b are mutually exclusive events—say $a = \{1\}$ and $b = \{246\}$ in the example of the die. (In the algebraic theory mutually exclusive events correspond to "disjoint" elements of the Boolean algebra B, that is to elements

a and b for which $a \cap b = o$.) In this case the proportion of times that the join $a \cup b (= \{1246\}$ for the example) occurs is clearly the sum of the proportions associated with a and b separately. If an ace shows up one-sixth of the time and an even number half the time, then the proportion of times in which the top face is either an ace or an even number is $\frac{1}{6} + \frac{1}{2}$. It follows therefore that the function P cannot be completely arbitrary—it is necessary to subject it to the condition of additivity, that is to require that if $a \cap b = o$ then $P(a \cup b)$ should be equal to $P(a) + P(b)$.

We are now separated from the final definition of probability theory only by a seemingly petty (but in fact very important) technicality. If $P(a)$ is an additive function of the sort just described on a Boolean σ-algebra B, and if $a_1, a_2, \cdots, a_n$ is any finite set of pairwise disjoint elements of B (this means that for $i \neq j$, $a_i \cap a_j = o$) then it's easy to prove by mathematical induction that $P(a_1 \cup a_2 \cup \cdots \cup a_n) = P(a_1) + P(a_2) + \cdots + P(a_n)$. If however $a_1, a_2, a_3, \cdots$ is an infinite sequence of pairwise disjoint elements then it may or may not be true that $P(a_1 \cup a_2 \cup a_3 \cup \cdots) = P(a_1) + P(a_2) + P(a_3) + \cdots$. The general condition of countable (that is, finite or enumerably infinite) additivity is a further restriction on the probability measure P—a restriction without which modern probability theory could not function. It is a tenable point of view that our intuition demands infinite additivity just as much as finite additivity. At least however infinite additivity does not contradict any of our intuitive ideas and the theory built on it is sufficiently far developed to assert that the assumption is justified by its success. We shall therefore adopt this assumption as our final postulate.

Numerical probability is a measure function, that is a finite, nonnegative, and countably additive function P of elements in a Boolean σ-algebra B, such that if the null and unit elements of B are o and e respectively then $P(a) = 0$ is equivalent to $a = o$ and $P(a) = 1$ is equivalent to $a = e$.

In the next section we shall discuss a general method of constructing examples of probability measures.

4. Measure space. Let $\omega_j (j = 1, \cdots, 6)$ be the point on the real axis whose directed distance from the origin is j, and let Ω be the set whose elements are these six points. Consider the system B^* of all subsets of Ω. (The empty set o and the full set $e = \Omega$ are counted as belonging to B^*.) With any element a of B^* (that is, with any subset of Ω) we may associate the complementary element (set) consisting of exactly those points ω_j which do not belong to a. Similarly with any two subsets a and b of Ω we may associate their union (the set of points belonging to either a or b or both), and their intersection (the set of points belonging simultaneously to a and b). It is easy to verify that under the operations of complementation (a'), formation of unions ($a \cup b$), and formation of intersections ($a \cap b$), the system B^* forms a Boolean algebra, in fact, though somewhat vacuously, a σ-algebra. Suppose moreover that for each $j = 1, \cdots, 6$, p_j is a positive number such that $p_1 + \cdots + p_6 = 1$. Then we may define $P(a)$ for any subset a of Ω, to be the sum of those p_j whose ω_j belongs to a. Thus if

$a=\{135\}$ then $P(a)=p_1+p_3+p_5$; if $a=o$ then $P(a)=0$. The function P and the algebra B^* satisfy all the assumptions of probability theory and the reader has doubtless recognized that this B^* and P were implicit in our earlier discussion of the rolling die. It is often customary on philosophical and practical grounds to discuss only the case $p_1=\cdots=p_6=\frac{1}{6}$. We shall say a word about this special case later; for the moment it is sufficient to point out that any other choice of the p_j furnishes an equally acceptable probability structure and does in fact constitute the mathematical theory of some carefully loaded die.

The above example of a Boolean algebra can be generalized: we attempt next to obtain a similar but more geometrical example. For this purpose we again choose a set Ω, but, instead of a finite set, we choose a set with infinitely many points, in fact all the points of a continuum. To be specific let us choose for Ω the points ω of a square of unit area in the Cartesian plane. In analogy with the preceding example we consider the system B^* of all subsets of Ω and define complement, union, and intersection as before. Once more B^* is a Boolean σ-algebra; it is not however the one on which we shall base our probability theory. (It can be shown that it is not possible to define a probability measure P with the desired properties on B^*.) We shall instead consider a certain subsystem (sub-algebra) of B^*, constructed as follows:

We begin with the system R of all rectangles contained in Ω (where for the sake of definiteness we consider closed rectangles, that is sets consisting of the interior plus the perimeter of a rectangle). The system R is not closed under the Boolean operations: in general not even a finite (let alone a countably infinite) union or intersection of rectangles is itself a rectangle, and similarly the complement of a rectangle isn't one. We have therefore to enlarge the system R to a system R' including all complements and countable unions and intersections of elements of R. It turns out that even this is not enough: R' is still not a Boolean algebra, and the extension process has to be continued. If however the extension process is continued sufficiently (and this happens to mean transfinitely) often, we reach eventually a Boolean σ-algebra B of subsets of Ω. (The algebra B is important in analysis: sets of B are called the Borel sets of the square.)

We face next the task of defining P. For those familiar with the theory of Lebesgue measure it will suffice to say that we define $P(a)$, for each a in B, to be the Lebesgue measure of the set a. It is not difficult to get an intuitive idea of how P is defined. If a is a rectangle (that is an element of R) we define $P(a)$ to be the area of a. If a is an element of R' we proceed to determine $P(a)$ in accordance with the requirement of countable additivity. Thus for example if b is the complement of a rectangle a, we write $P(b)=1-P(a)$, and if b is the union of a finite or infinite sequence of disjoint rectangles $a_1, a_2, \cdots$ we define $P(b)=P(a_1)+P(a_2)+\cdots$. By repeating this extension process ad transfinitum we succeed eventually in defining $P(a)$ for every a in B.

There is an objection to the construction just described. If the set a consists of a single point then it is intuitively obvious (and follows easily from the rigorous definition of P) that $P(a)$ (=the area of a) is zero. More generally if a

consists of any finite or enumerably infinite set of points we still have $P(a)=0$, and it is even possible (if for example a is a line segment) to have $P(a)=0$ for sets a containing uncountably many points. This definitely contradicts our explicitly formulated axiom that $P(a)=0$ should happen if and only if $a=o$. The customary way to get around this difficulty is by redefining the notion of equality that occurs in the equation $a=o$. It is proposed that we agree to consider as identical two subsets of Ω whose difference has probability zero. (In technical language, we consider, instead of the sets a, equivalence classes of sets modulo the class of sets of probability zero.) Through this agreement we are committed in particular to identifying any set of probability zero with the empty set o, and it follows therefore that in the reduced algebra B (that is, the algebra obtained from B by making the suggested identifications) all the axioms of probability are valid.

The long and tortuous process just described is very general. If Ω is any space (such as an interval or a cube) on a certain σ-algebra B of subsets of which a countably additive measure P is defined (such as length or volume), subject only to the restriction that the measure of all Ω is equal to 1, we obtain from B and P a system satisfying all the axioms of probability theory by the process of identification according to sets of measure zero. Thus there are as many probability systems as there are examples of "measure spaces."

The reason for the introduction of measure spaces into a discussion of probability theory is not merely to give examples. It can in fact be shown that the two theories (measure and probability) are coextensive. More precisely:

If B is any Boolean σ-algebra and P a probability measure on B, then there exists a measure space Ω such that the system B is abstractly identical with an algebra of subsets of Ω reduced by identification according to sets of measure zero, and the value of P for any event a is identical with the values of the measure for the corresponding subsets of Ω.

Hence measure is probability and probability is measure and, in virtue of the theorem just stated, the entire classical theory of measure and integration may be and has been carried over and used to give rigorous proofs of probability theorems.

5. Measure vs. probability. Having discussed the extent to which probability and measure are the same, we now dedicate a few words to describing the extent to which they are different. One feature that differentiates the two theories is that in the general theory of measure it is usual to admit the possibility that the measure of the entire space is infinite. This possibility is not admissable in probability theory. As long, however, as the measure of the whole space is finite it is always possible to introduce a scale factor which makes it equal to 1, and hence it is always possible to think of it (even if somewhat artificially) as a "probability space." Thus for example the language and notation of probability may be and have been used in such seemingly widely separated parts of mathematics as ergodic theory, topological groups, and integral geometry.

Even however if the infinite case is ruled out, it is a conspicuous fact that most theorems in which the word measure is used (rather than the word probability) have a very different appearance from the theorems of probability theory. The best way to explain the difference between measure and probability is to liken it to the difference between analytic and synthetic geometry. It isn't stretching a point too far to say that the representation of a probability algebra by a measure space is similar to the introduction of coordinates into geometry. Synthetic and analytic geometry are of course abstractly identical in the sense that any theorem in the one domain may be stated and proved in the language and machinery of the other—may be, but isn't. The theorems in the two fields differ in their intuitive content. It is natural to discuss linear transformations in analytic geometry and the nine point circle in synthetic geometry—and even though the interchange is possible, it isn't desired. The abstract identity of the two fields is however an extremely useful fact, exploited mostly by the synthetic side which often finds it convenient to lean on the analytic crutch. Similarly, probability is measure, and research in the field would be very greatly hampered if we were not permitted to use this analytic crutch—but the notions suggested by probability, the notions which are important and intuitive and natural inside the field, appear sometimes extremely special and artificial in the frame work of general measure theory.

In this section and the preceding ones we have treated axiomatization and coordinatization. We proceed now to development. In the following sections we shall define the basic concepts of probability theory, and discuss in particular those which serve in the sense described above to give to probability its distinguishing flavor.

6. Independent events. In order to motivate the definitions of the concepts to be studied in the sequel we return to the example of the die. For simplicity we make the classical assumption that any two faces are equally likely to turn up and that consequently the probability of any particular face showing is $\frac{1}{6}$. Consider the events $a = \{246\}$ and $b = \{12\}$. The first notion we want to introduce, the notion of conditional probability, can be used to answer such questions as these: "what is the probability of a when b is known to have occurred?" In the case of the example: if we know that v is less than 3, what can we say about the probability that v is even? The adjective "conditional" is clearly called for in the answer to a question of this type: we are evaluating probabilities subject to certain preassigned conditions.

To get a clue to the answer consider first the event $c = \{2\}$ and ask for the conditional probability of a, given that c has already occurred. The intuitive answer is perfectly clear here, and is independent as it happens of any such numerical assumptions as the equal likelihood of the faces. If v is known to be 2 then v is certainly even, and the probability must be 1. What made the answer easy was the fact that c implied a. The general question of conditional probability asks us to evaluate the extent (measured by a numerical probability or propor-

tion) to which the given event b implies the unknown event a. Phrased in this way the question almost suggests its own answer: the extent to which b is contained in a can be measured by the extent to which a and b are likely to occur simultaneously, that is by $P(a \cap b)$. Almost—not quite. The trouble is that $P(a \cap b)$ may be very small for two reasons: one is that not much of b is contained in a, and the other is that there isn't very much of b altogether. In other words it isn't merely the absolute size of $a \cap b$ that matters: it's the relation or proportion of this size to the size of b that's relevant.

We are led therefore to define the conditional probability of a, given that b has occurred, in symbols $P_b(a)$, as the ratio $P(a \cap b)/P(b)$. For $a = \{246\}$ and $c = \{2\}$ this gives the answer we derived earlier, $P_c(a) = 1$; for $a = \{246\}$ and $b = \{12\}$ we get the rather reasonable figure $P_b(a) = \frac{1}{2}$. In other words if it's known that v is either 1 or 2 then v is even or odd (that is equal to 1 or equal to 2) each with probability $\frac{1}{2}$.

Consider now the following two questions: "b happened, what is the chance of a?" and simply "what is the chance of a?." The answers of course are $P_b(a)$ and $P(a)$ respectively. It might happen, and does in the example given above, that the two answers are the same, that in other words knowledge of b contributes nothing to our knowledge of the probability of a. It seems natural in this situation to use the word "independent": the probability distribution of a is independent of the knowledge of b. This motivates the precise definition: two events a and b are independent if $P_b(a) = P(a)$. The definition is transformed into its more usual form and at the same time gains in symmetry if we recall the definition of $P_b(a)$. In symmetric form: a and b are independent in the sense of probability (statistically or stochastically independent) if and only if $P(a \cap b) = P(a)P(b)$.

7. Repeated trials. Suppose next that we wish to make two independent trials of the same experiment—say, for example, to roll an honest die twice in succession. We shall presently exploit the precise definition of independence to clarify the notion of independent trials; first however it's worth while to remark on the intuitive content of the concept. Suppose that in a crude attempt to even things up we resolve on the following procedure: if the first die shows an even number we choose for the second experiment a die on which all the numbers are odd, and vice versa. The two experiments are not independent of each other in this case: whereas the a priori probability of getting an even number with the second die is $\frac{1}{2}$, the conditional probability of getting an even number with the second die, given that the first one showed an odd number, is one. We say that the two experiments are performed independently of each other only if the conditions under which the second experiment is to be performed are unaffected by the outcome of the first experiment.

If an experiment consists of two rolls of a die we don't expect the reported outcome of the experiment to be a number v, but rather a pair of numbers (v_1, v_2). The measure space Ω associated with the two-fold experiment consists

not of 6 but of 36 points. (It is convenient to imagine these points laid out along the regular pattern of a 6×6 square.) The problem is to determine how the probability is distributed among these points. For a clue to the answer consider the events $a=$"$v_1<3$" and $b=$"$v_2<4$." We have $P(a)=\frac{1}{3}$ and $P(b)=\frac{1}{2}$; hence if we interpret the independence of the trials to mean the independence of any two events such as a and b we should have $P(a\cap b)=\frac{1}{6}$. If in the suggested diagram for the measure space associated with this discussion we encircle the points belonging to $a\cap b$ we get the following figure.

v_2 \ v_1	1	2	3	4	5	6
1	⊙	⊙	·	·	·	·
2	⊙	⊙	·	·	·	·
3	⊙	⊙	·	·	·	·
4	·	·	·	·	·	·
5	·	·	·	·	·	·
6	·	·	·	·	·	·

We see therefore that the formula $P(a\cap b)=P(a)P(b)$ appears analogous to the fact that the area of a rectangle is the product of the lengths of its sides.

We say therefore, if the analytic description of an experiment is given by a measure space Ω with a Boolean σ-algebra B of subsets on which a probability measure P is defined, that the analytic description of the experiment consisting of two independent trials of the given experiment is as follows. The space of points ω is replaced by the space of pairs of points (ω_1, ω_2) (the so called product space $\Omega\times\Omega$), B is replaced by the Boolean σ-algebra generated by the "rectangular" sets of the form $\{\omega_1$ is in a_1, ω_2 is in $a_2\}$ where a_1 and a_2 belong to B, and the probability measure on this space of pairs is determined by the requirement that its value for rectangular sets of the kind described should be given by the product $P(a_1)P(a_2)$. The ideas involved in this procedure are not essentially original nor characteristic of probability theory: they are the same as the ideas involved in defining the area of plane sets in terms of the length of linear sets. There is of course a theorem hidden in this definition—a theorem which asserts that a probability measure satisfying the stated product requirement indeed exists and is in fact uniquely determined by this requirement.

What we can do once, we can do again. Just as two repetitions of an experiment gave rise to ordered pairs (ω_1, ω_2), similarly any finite number of repetitions (say n) give rise to the space of ordered n-tuples $(\omega_1, \omega_2, \cdots, \omega_n)$, with a multiplicatively determined probability measure. The procedure can be extended also to infinity: the analytic model of an infinite sequence of independent repetitions of an experiment is a measure space Ω whose points ω are infinite sequences $\{\omega_1, \omega_2, \omega_3, \cdots\}$. Even if an actually infinite sequence of repetitions of an experiment is practically unthinkable, there is a point in considering the infinite dimensional space Ω. The point is that many probability statements are asser-

tions concerning what happens in the long run—assertions which can be made precise only by carefully formulated theorems concerning limits. Hence even if practice yields only approximations to infinity, it is the infinite sequence space Ω that is the touchstone whereby the mathematical theory of probability can be tested against our intuitive ideas. The first and most important such long run statement is described in the following paragraphs.

Suppose that an experiment is capable of producing an event a with probability p, and suppose that an infinite sequence of independent trials of this experiment is performed. We consider therefore the space of all sequences $\omega = \{\omega_1, \omega_2, \omega_3, \cdots\}$ where for each n, ω_n may or may not belong to a. Once the experiments have been performed so that we are given a particular point ω we may start asking numerical questions. We may ask for example: out of the first n trials of the basic experiment how many resulted in a? This means: out of the first n coordinates $\omega_1, \omega_2, \cdots, \omega_n$ of ω how many belong to a? The answer to this question depends obviously on n and just as essentially on the particular sequence ω—let us denote it by $m_n(\omega)$.

Now what does out intuition say? The usual statement (one which we have already exploited in our heuristic derivation of the notion of probability) is that the ratio of the number of successes to the total number of trials should be approximately equal to the probability of the event being tested. In our notation this seems to mean that for large n the ratio $m_n(\omega)/n$ should be close to the constant $p = P(a)$. The question arises: for which ω's should this be true? Not surely for all of them. For the sequence space Ω contains sequences none of whose coordinates belong to a, and for such a sequence ω, $m_n(\omega)$ is zero for all n. The best that we have a right to demand is that the ω's for which our statement is not true should be equivalent to the empty set of ω's in the sense of probability—that is that their totality should have probability zero. And this is true.

To sum up: we have just derived the statement (not the proof) of the most important special case of the so called strong law of large numbers. In mathematical language the assertion of this law is that as $n \to \infty$, $\lim m_n(\omega)/n$ exists and is equal to $p(=P(a))$ except for a set of ω's of measure zero. In more classical terms: it is almost certain that the "success ratios" converge to the probability of the event being tested.

8. Random variables. In order to gain a more thorough understanding of the law of large numbers and at the same time to introduce the language in which most of the theorems of probability theory are stated, we proceed to discuss the notion of a random variable.

"A random variable is a quantity whose values are determined by chance." What does that mean? The word "quantity" is meant to suggest magnitude—numerical magnitude. Ever since rigor has come to be demanded in mathematical definitions it has been recognized that the word "variable," particularly a variable whose values are "determined" somehow or other, means in precise language a function. Accordingly a random variable is a function: a function

whose numerical values are determined by chance. This means in other words that a random variable is a function attached to an experiment—once the experiment has been performed the value of the function is known. The spatial model of probability is extremely well adapted to making this notion still more precise. If the analytic correspondent of an experiment is a measure space Ω then any possible outcome of the experiment is by definition represented by a point ω in this space. Hence a function of outcomes is a function of ω's: a random variable is a real valued function defined on a probability space Ω.

The preceding sentence does not yet constitute our final definition of a random variable. For suppose that $x=x(\omega)$ is a function on the space Ω. We shall call x a random variable only if probability questions concerning the values of x can be answered. An example of such a question is: what is the probability that x is between α and β? In measure theoretic language: what is the measure of the set of those ω's for which the inequality $\alpha \leqq x(\omega) \leqq \beta$ is satisfied? In order for such questions to be answerable it is necessary and sufficient that the sets that occur in them belong to the basic σ-algebra B of Ω. A function $x(\omega)$ for which this is true for every interval (α, β) is called "measurable." Accordingly we make the following definition:

A random variable is a measurable function defined on a measure space with total measure 1.

Instances of random variables can be found even in that part of our discussion which preceded their definition. The quantity v associated with the rolling die is an example, as are also the quantities v_1 and v_2 associated with the two fold repetition of this experiment. To obtain some further examples, consider any fixed event a which may result from an experiment and let the random variable x be the number of times that a actually occurs. If the experiment is performed only once then x has only two possible values: 1 if a occurs and 0 otherwise. More generally if the experiment is repeated n times the random variable x becomes the function $m_n(\omega)$ introduced in the discussion of the law of large numbers.

9. Expectation, variance, and distribution. Let us consider in detail the random variable v associated with an honest die. The possible values of v are the first six positive integers. The arithmetic mean of these values, that is the number $(1+\cdots+6)/6$, is of considerable interest in probability theory. It is called the average, or mean value, or expectation of the random variable v and it is denoted by $E(v)$. If the die is loaded so that the probability p_j associated with j is not necessarily $\frac{1}{6}$ then the arithmetic mean is replaced by a weighted average: in this case $E(v)=1\cdot p_1+\cdots+6\cdot p_6$. It is well known that the analogs of such weighted sums in cases where the number of values of the function (random variable) need not be finite are given by integrals. The kind of integral that enters into probability theory is similar in every detail to the Lebesgue integral and we shall not reproduce its definition here.

If the measurable function $x(\omega)$ is integrable then its expectation $E(x)$ is by definition the value of its integral extended over the entire domain Ω.

As a useful though extremely special case we mention that if x is a counting variable of the sort mentioned in the preceding paragraph ($x=1$ if a certain even a occurs and $x=0$ otherwise) then $E(x)=P(a)$.

It is obviously of interest to ask not only what is the expected value of a random variable x but also how closely the values of x are clustered about its expected value. The customary measure of clustering of a random variable x is one inspired by the method of least squares and called the "variance" or "dispersion" of x.

The variance of x is the expression $\sigma^2(x)=E(x-\alpha)^2$, where $\alpha=E(x)$.

(The square root of the variance is called the "standard deviation.") In words: take the square of the deviation of x from its expected value α, and use the sum (weighted sum, integral) of these squared deviations as a measure of clustering. Since a sum of squares vanishes only if each term does, the vanishing of the variance indicates that x is identically equal to its expected value (except perhaps for a set of probability zero). In general, the smaller the variance the closer the values of x lie to $E(x)$.

Such numbers as $E(x)$ and $\sigma^2(x)$ yield partial information about the distribution of the values of x. Complete information would mean an answer to every question of the form "what is the probability that x lies in the interval (α, β)?" In order to deal with such questions we introduce the notion of distribution function.

The distribution function $F_x(\lambda)$ of a random variable x is a function of a real variable λ defined for each λ to be the probability that $x<\lambda$.

These functions can be used to answer every probability question concerning random variables; for example the expression $F_x(\beta)-F_x(\alpha)$ represents the probability that x belong to the (half open) interval $\alpha\leqq x<\beta$, and the Stieltjes integrals $\int_{-\infty}^{\infty}\lambda dF_x(\lambda)$ and $\int_{-\infty}^{\infty}\{\lambda-E(x)\}^2dF_x(\lambda)$ represent the expectation and variance of x respectively. Distribution functions are useful because being comparatively simple real functions of real variables they are amenable to treatment by the methods of classical analysis. It is the whole purpose of a large part of probability theory to find the distribution functions of certain random variables.

10. Independent variables. Let us consider next two random variables x and y which are comparable in the sense that they are both represented by measurable functions on the same measure space Ω, so that $x=x(\omega)$ and $y=y(\omega)$. It is easy to see that the function $E(x)$, being defined by an integral, is homogeneous of degree 1 and additive, that is $E(\lambda x)=\lambda E(x)$ for every real constant λ and $E(x+y)=E(x)+E(y)$. Similarly the variance $\sigma^2(x)$ is homogeneous of degree 2, that is $\sigma^2(\lambda x)=\lambda^2\sigma^2(x)$. One way to prove this latter fact is to make use of the following identity connecting σ^2 and E:

$$\sigma^2(x) = E(x)^2 - E^2(x), \tag{1}$$

(where for later convenience we write $E(x)^2$ for $E(x^2)$ and $E^2(x)$ for $\{E(x)\}^2$). This identity in turn follows from the definition of σ^2. Since $\sigma^2(x)=E(x-\alpha)^2$

where $\alpha = E(x)$, we have $\sigma^2(x) = E(x^2 - 2\alpha x + \alpha^2) = E(x^2) - 2\alpha E(x) + \alpha^2$. (We used here the fact that the expected value of a constant is equal to that constant.) The identity (1) follows by substituting for α its value $E(x)$. Letting the formalism guide us we may inquire whether σ^2 is additive, that is whether or not the identity

$$\sigma^2(x + y) = \sigma^2(x) + \sigma^2(y) \tag{2}$$

is valid. The answer in general is no. In order to investigate conditions under which (2) is true we proceed to a brief discussion of some possible relations between pairs of random variables.

Let a and b be two independent events and let x and y be the associated counting random variables (so that x for example is 1 if and only if a occurs and $x = 0$ otherwise). The product random variable xy in this case can be equal to 1 if and only if both a and b occur, so that xy is the counting variable of $a \cap b$. Since $E(x) = P(a)$, $E(y) = P(b)$, and similarly $E(xy) = P(a \cap b)$, we have in this special case

$$E(xy) = E(x)E(y). \tag{3}$$

The validity of this formula is sufficiently important in the applications of probability to bear a name of its own: two random variables, not necessarily the counting variables of a pair of independent events, satisfying it are called "uncorrelated." The reason for the terminology is that the coefficient of correlation $r = r(x, y)$ of two random variables x and y is defined by $r = \{E(xy) - E(x)E(y)\}/\sigma^2(x)\sigma^2(y)$; this coefficient vanishes if and only if (3) holds.

It is now easy to state the facts concerning the formula (2): it is valid if and only if (3) is. In other words the variance is additive for a pair of random variables if and only if the expectation is multiplicative, that is if and only if they are uncorrelated. For the proof we merely expand the left member of (2), thus:

$$\begin{aligned}\sigma^2(x + y) &= E(x + y)^2 - E^2(x + y) \\ &= \{E(x)^2 - 2E(xy) + E(y)^2\} - \{E^2(x) - 2E(x)E(y) + E^2(y)\} \\ &= \sigma^2(x) + \sigma^2(y) - 2\{E(xy) - E(x)E(y)\}.\end{aligned}$$

Let us now return to the pair of counting variables x and y associated with two independent events a and b. Because of the independence of a and b, any probability statement concerning y is unaffected by our knowledge of ignorance of the value of x. More precisely, any two events defined by x and y, for example the events "$x = 0$" and "$y = 1$," are independent. If in general any two events by two random variables x and y respectively, that is any two events defined by inequalities of the form $\alpha \leqq x \leqq \beta$ and $\gamma \leqq y \leqq \delta$, are independent events, no matter what α, β, γ and δ are, we say that x and y are independent random variables. It is not too difficult to generalize what we proved about the special case of counting variables: for independent random variables the expectation, if it exists, is multiplicative and consequently the variance is additive. In still

other words: independence implies absence of correlation—a proposition which certainly sounds natural enough.

One word of caution before we leave this brief introduction to the notion of independence for random variables. What we defined was the independence of two random variables. It would be natural to try to define the independence of a finite or infinite sequence of random variables $x_1, x_2, \cdots$, by the requirement that any pair be independent. Natural, but as it happens, not very useful. The correct definition replaces two-term products by many-term products in the following way.

The random variables $x_1, x_2, \cdots$, are independent if the probability of the simultaneous occurrence of any finite number of the events defined by $\alpha_n \leqq x_n \leqq \beta_n$ is the product of the separate probabilities, no matter what real constants the α's and β's are.

It is easy to construct examples to show that this notion is indeed different from the notion of pairwise independence.

11. Law of large numbers. We are now in a position to reformulate and generalize the strong law of large numbers in terms of random variables. Let the sequence space of points $\omega = \{\omega_1, \omega_2, \cdots\}$ be the analytic model of the infinite repetition of an experiment one of whose possible outcomes is the event a. Let a_n be the event "ω_n belongs to a" or equivalently the event "the nth experiment results in a," and let $x_n = x_n(\omega)$ be the counting variable associated with a_n. In this context that means that $x_n(\omega)$ has the value 1 for all those sequences $\omega\{\omega_1, \omega_2, \cdots\}$ for which the nth coordinate ω_n belongs to a, and $x_n(\omega)$ has the value 0 otherwise. What significance has the sum $x_1 + \cdots + x_n$? Since a particular term x_j contributes one unit to this sum if and only if the jth experiment results in a, it is clear that the value of the sum, for any sequence ω, is the number of those coordinates among the first n coordinates of ω which do belong to a. But this is exactly the function we denoted above by $m_n(\omega)$. Hence our version of the law of large numbers is equivalent to the assertion that the averages $(x_1 + \cdots + x_n)/n$ converge (except possibly for a set of ω's of probability zero) to the constant $p = P(a)$. For the generalization of this result that we are about to formulate it is worth while to observe that $p = E(x_n)$ is also equal to the common value of the expectations of the x's.

The sequence of random variables $x_1, x_2, \cdots$ has two important properties which are sufficient to ensure the validity of the law of large numbers. One of these properties is independence. It follows very easily from the fact that the experiments yielding the values of the various x's are independently performed, that the variables $x_1, x_2, \cdots$ are indeed independent. The other essential property of the sequence is usually expressed by the statement that the random variables x_n all have the same distribution. The definition of this concept is as follows.

Two random variables x and y have the same distribution if for every interval (α, β) the probabilities of the two events $\alpha \leqq x \leqq \beta$ and $\alpha \leqq y \leqq \beta$ are equal, or equivalently if the distribution functions $F_x(\lambda)$ and $F_y(\lambda)$ are identical.

In our particular case it is the fact that the probability that ω_n belong to a is the same for all n (namely $P(a)$) that implies that the x_n all have the same distribution. That independence and equidistribution are indeed the crucial hypotheses for the law of large numbers is shown by the following general formulation of that law.

If $x_1, x_2, \cdots$ is a sequence of independent random variables with the same distribution, and if the expectations $E(x_n)$ exist and have the value α (necessarily the same for all n) then the averages $x_1+\cdots+x_n/n$ converge as $n\to\infty$ (except perhaps on a set of probability zero) to the constant α.

12. Central limit theorem. Sums (such as $x_1+\cdots+x_n$) of independent random variables with the same distribution occur very often in probability theory. It is of considerable practical importance to investigate the precise distribution of such sums and if possible the limiting behavior of these distributions. We assume concerning the x's that their expectations and variances both exist and write $E(x_j)=\alpha$, $\sigma^2(x_j)=\beta$. It follows from the independence and equidistribution of the x's that $E(x_1+\cdots+x_n)=n\alpha$ and $\sigma^2(x_1+\cdots+x_n)=n\beta$. At first sight this seems like a discouraging phenomenon: if both the expectation and the variance become infinite, how can we expect a reasonable asymptotic behavior from the much more delicate distribution function? But the way out of the difficulty is easy: by a translation and a change of scale (different to be sure for each n) it is possible to normalize the sum $x_1+\cdots+x_n$ so that its expectation is 0 and its variance 1 for every positive integer n. To get the expectation to be 0 we merely subtract its actual value, $n\alpha$, from the sum—the additivity of the expectation ensures the desired result. To get the variance to be 1 we divide by a constant factor. It is important to recall that the variance is homogeneous of degree 2, so that the constant factor will be not $n\beta$ but $\sqrt{n\beta}$. We arrive thus at the normalized sums

$$\frac{x_1+\cdots+x_n-n\alpha}{\sqrt{n\beta}}$$

and inquire again after the distribution function of this random variable and the limit of such distribution functions. The answer here is known and is embodied in the so called central limit theorem (or Laplace-Liapounoff theorem) stated as follows.

If $x_1, x_2, \cdots$ is a sequence of independent random variables with the same distribution, expectation α, and variance β, then the distribution functions of the modified sums $(x_1+\cdots+x_n-n\alpha)/\sqrt{n\beta}$ converge as $n\to\infty$ to a fixed distribution function, the same no matter what the original distribution of the x's is. In more detail, the limit as $n\to\infty$ of the probability of the event defined by the inequality

$$\frac{x_1+\cdots+x_n-n\alpha}{\sqrt{n\beta}}<\lambda$$

exists and is equal to

$$G(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda}e^{-u^2/2}du.$$

The distribution function $G(\lambda)$ is called the Gaussian or normal distribution.

With this statement we end our discussion of the development of probability theory and turn to a few remarks connected with the problem of application.

13. Determination of initial probabilities. When the mathematician announces that the probability of an event is a certain number, he is immediately faced with two questions. First the practical man asks what is the practical meaning of a probability statement? How should one act on it? If the mathematician succeeds in answering this question then the philosopher wants to know the reason for the answer. What establishes the connection between mathematical theory and practice? Our remarks in what follows will bear on these very old and very difficult questions only incidentally—they are dedicated mainly to a smaller problem of the theory, but one which frequently worries the layman.

The problem is how the probability of concretely given events is really defined. It is all very well to talk about Boolean algebras and measure theory, but what is the probability that a coin will fall heads up? What the layman realizes and what we now wish to emphasize is that the mathematician has not answered any such questions. He cannot. He can no more say that the probability of obtaining two heads in succession with a coin is $\frac{1}{4}$ than he can say that the volume of a cube is 8. The volume of a cube is given by a formula. If the hypotheses under which the formula applies are verified and if the variables entering into the formula are given specific values then the volume of a cube can be calculated. In exactly the same sense the mathematical theory of probability is a collection of formulae which enable us to calculate certain probabilities assuming that certain other ones are given. If we know that the probability of obtaining heads with a certain coin is $\frac{1}{2}$ and if we know that two successive tosses of the coin were performed independently then we can assert that the probability of getting two heads is $\frac{1}{4}$.

Despite the fact that probability theory shares with all other mathematical theories its inability to state a conclusion without hypotheses, the above answer to the layman's question will probably seem unsatisfactory to many readers. There must be some reason why most people believe that the probability of heads is $\frac{1}{2}$. It is often even proved. The usual proof is based on symmetry arguments, or equivalently on the principle of sufficient reason. (Why should heads have any greater likelihood of appearing than tails?) Do these proofs have any mathematical validity?

The answer is definitely yes. In some cases it is more pleasing to the intuition or more convenient for practice to formulate our hypotheses purely qualitatively. In almost all such cases the hypotheses take the form of invariance—the probabilities entering into the problem are required to be invariant under a certain group of transformations. It often turns out then that an existence and uniqueness theorem is true, that is it can be proved that there exists one and only one probability measure satisfying the stated hypotheses. Theorems of this type are certainly a part, an increasingly important part, of the theory of proba-

bility, and as long as their hypotheses are clearly formulated and recognized as hypotheses, the professional mathematician is the last person to sneer at them. Their advantage at the level of elementary pedagogy seems to lie in the fact that the statement "heads and tails are equally likely" is easier to grasp intuitively than the statement "the probability of heads is $\frac{1}{2}$."

We see thus that a mathematical statement on probability has to have certain either explicitly or implicitly given probabilities to begin with. In practice the physicist (or actuary, or anyone else interested in applying the theory) obtains these initial numbers experimentally. If he wants to know what is the probability of a coin falling heads up, he tosses the coin a large number of times and then uses the law of large numbers to assure himself that he may use the obtained frequency ratio as an approximation to the correct value of the probability. Or he may observe that the values of a random variable are obtained as the sum of a large number of independent variables each with a negligible variance and thus be led to introduce the normal distribution. Such approximative procedures are of course common to all parts of applied mathematics.

14. Conclusion. Our exposition is finished. If the reader has been patient enough to read this far he may be curious enough to read farther. Our scanty bibliography will furnish a basis for such reading. For certainly not all probability theory is contained in this paper, nor as yet in any collection of books or papers. There is still much room in the field for the exercise of the analytic ingenuity and abstract generality of both classical and modern mathematics. If this paper will be instrumental in persuading mathematicians that probability is mathematics, and in causing some to look into the subject more deeply than they had previously thought worth while, it will have more than accomplished its purpose.

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NON-ANALYTIC FUNCTIONS

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The triumph of the theory of analytic functions lies in the fact that it has wide applications not only in other branches of mathematics but also in many physical investigations. In regard to the latter, it is possible merely because many physical quantities are distributed like the values of a harmonic function,

the real part of an analytic function. Since the likeness is actually approximate, we are compelled to ask the question whether such applications may not lead to serious mistakes and if not, what is the limit of the errors?

It is the purpose of this note to investigate non-analytic functions [1] which are "approximately analytic" and find the necessary modifications for some fundamental theorems in the theory of analytic functions. This furnishes an indirect answer of the question suggested.

1. Functions of non-analyticity r. If $f(z)=X(x, y)+i\,Y(x, y)$ is an analytic function of z in a region R, it has, at each point of R, a unique derivative

$$f'(z) = \lim_{\delta \to 0} \frac{f(z+\delta) - f(z)}{\delta}.$$

The derivative will not exist if $f(z)$ is non-analytic. We have, however, the following theorem.

THEOREM 1. *Let $X(x, y)$ and $Y(x, y)$ be continuous and have continuous partial derivatives of the first order near $z_0=x_0+iy_0$ and let*

$$w(\lambda) = u(\lambda) + iv(\lambda) = \lim_{\delta \to 0}^{\lambda} \frac{f(z_0+\delta) - f(z_0)}{\delta}.$$

where $\lim_{\delta \to 0}^{\lambda}$ denotes that $\delta \to 0$ along a line of slope λ. Then the point $w(\lambda)$ lies on the circle [2]

$$[u - \tfrac{1}{2}(X_x^0 + Y_y^0)]^2 + [v - \tfrac{1}{2}(Y_x^0 - X_y^0)]^2 = r^2(z_0) \tag{1}$$

where

$$r(z_0) = \tfrac{1}{2}[(X_x^0 - Y_y^0)^2 + (Y_x^0 + X_y^0)^2]^{1/2}$$

and

$$X_x^0 = \left[\frac{\partial}{\partial x} X(x, y)\right]_{x=x_0, y=y_0} \quad \text{etc.}$$

Proof. Let $\delta=h+ki$. Then, by law of the mean,

$$\frac{f(z_0+\delta) - f(z_0)}{\delta} = \frac{X(x_0+h, y_0+k) - X(x_0, y_0) + i(Y(x_0+h, y_0+k) - Y(x_0, y_0))}{h+ki}$$

$$= \frac{hX_x^{\theta} + kX_y^{\theta} + i(hY_x^{\theta'} + kY_y^{\theta'})}{h+ki}$$

where $0<\theta<1$, $0<\theta'<1$ and

$$X_x^{\theta} = X_x(x_0+\theta h, y_0+\theta k), \qquad Y_x^{\theta'} = Y_x(x_0+\theta' h, y_0+\theta' k), \quad \text{etc.}$$

Let $k/h=\lambda$, then

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} = \frac{X_x'+\lambda X_y'+i(Y_x'+\lambda Y_y')}{1+\lambda i}$$

$$= \frac{1}{1+\lambda^2}[X_x'+\lambda(X_y'+Y_x')+\lambda^2 Y_y'] + \frac{i}{1+\lambda^2}[Y_x'+\lambda(Y_y'-X_x')-\lambda^2 X_y'].$$

Since X_x, X_y, Y_x and Y_y are continuous near (x_0, y_0), we have

$$u(\lambda) = \frac{1}{1+\lambda^2}[X_x^0+\lambda(X_y^0+Y_x^0)+\lambda^2 Y_y^0],$$

$$v(\lambda) = \frac{1}{1+\lambda^2}[Y_x^0+\lambda(Y_y^0-X_x^0)-\lambda^2 X_y^0].$$

The theorem follows by a direct computation.

We define (1) as the *derivative circle* and its center the *derivative* of $f(z)$ at $z=z_0$. We write [3]

$$f'(z) = \tfrac{1}{2}(X_x+Y_y)+\tfrac{1}{2}i\,(Y_x-X_y).$$

It is interesting to note that if $r(z)=0$, the Riemann-Cauchy differential equations are satisfied and the function $f(z)$ is analytic. Thus we may define $r(z_0)$ as the *non-analyticity* of $f(z)$ at $z=z_0$, and its least upper bound in R, the non-analyticity of $f(z)$ in R.

THEOREM 2. *If $f(z)$ is of non-analyticity $r\geqq 0$ at $z=z_0$, then for any given $\epsilon>0$, we can find $\delta_0>0$ such that*

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right| = r+\epsilon\eta \tag{2}$$

provided that $0<|z-z_0|\leqq\delta_0$ where $-1\leqq\eta\leqq 1$.

Proof. As in the proof of Theorem 1,

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} = \frac{1}{1+\lambda^2}[X_x'+\lambda(X_y'+Y_x')+\lambda^2 Y_y'] + \frac{i}{1+\lambda^2}[Y_x'+\lambda(Y_y'-X_x')-\lambda^2 X_y'].$$

Since X_x, X_y, Y_x and Y_y are continuous, having given $\epsilon>0$ we can find $\delta_0>0$ such that

$$|X_x'-X_x^0|<\frac{\epsilon}{4}, \quad \textit{etc.},$$

provided that $|\delta| \leqq \delta_0$. Using the notations $u(\lambda)$ and $v(\lambda)$ of theorem 1, we have, then,

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} = u(\lambda) + iv(\lambda) + \eta'\epsilon$$

where $|\eta'| \leqq \frac{1}{2}(1+|\lambda|)^2/(1+\lambda^2) \leqq 1$. Therefore

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} - f'(z_0) = [u(\lambda) - \tfrac{1}{2}(X_x^0 + Y_y^0)] + i[v(\lambda) - \tfrac{1}{2}(Y_x^0 - X_y^0)] + \eta'\epsilon$$

$$= \xi r + \eta'\epsilon = \xi(r + \eta'\xi^{-1}\epsilon)$$

where $|\xi| = 1$, by theorem 1.

The theorem follows immediately, since

$$r + \epsilon \geqq |r + \eta'\xi^{-1}\epsilon| \geqq r - \epsilon.$$

In the preceding treatment, we have assumed that $X(x, y)$ and $Y(x, y)$ have continuous partial derivatives of the first order. It is better to generalize our definitions as follows.

DEFINITIONS. Let $f(z) = X(x, y) + i\,Y(x, y)$ be continuous in the region R and $X(x, y)$ and $Y(x, y)$ have partial derivatives of the first order in R. Let

$$f'(z) = \tfrac{1}{2}(X_x + Y_y) + \tfrac{1}{2}i\,(Y_x - X_y). \tag{3}$$

Let z_0 be a point in R. If for any $\epsilon > 0$ we can find $\delta_0 > 0$, such that

$$|f(z_0+\delta) - f(z_0) - \delta f'(z_0)| \leqq |\delta|\,(s+\epsilon), \qquad s \geqq 0$$

provided that $|\delta| \leqq \delta_0$, then $f(z)$ is called a function of *non-analyticity* $\leqq s$ at z. The greatest lower bound of s is called the *non-analyticity* of $f(z)$ at $z = z_0$. We denote it by $r(z_0)$. The least upper bound of $r(z)$ in R is called the non-analyticity of $f(z)$ in R.

Evidently we may put $s = r(z_0)$.

By the two-dimensional form of the modified Heine-Borel theorem,* we have

THEOREM 3. *Let $f(z)$ be of non-analyticity r in a region containing the closed region R. Then given $\epsilon > 0$ we can divide R into a finite number of parts (squares with sides parallel to the axes and their interiors or portions of such squares which do not exist when R is a square) such that inside or on the boundary of any part there is one point z_0 such that the inequality*

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leqq (r+\epsilon)(z - z_0)$$

is satisfied by all points z inside or on the boundary of that part.

* See Whittaker and Watson, A Course of Modern Analysis, first edition. §3.6.

From the definitions we derive immediately the following rules

THEOREM 4. *Let* $f_x(z)=X_x+iY_x, f_y(z)=X_y+iY_y$, *then*

(a) $$f'(z) = \tfrac{1}{2}(f_x - if_y),$$

(b) $$(f(z) + \phi(z))' = f'(z) + \phi'(z),$$

(c) $$(f(z)\phi(z))' = f(z)\phi'(z) + f'(z)\phi(z).$$

THEOREM 5. *Let* $f(z)$ *and* $\phi(z)$ *be functions of non-analyticity* r_1 *and* r_2 *respectively at* $z=z_0$ *(in a region containing the closed region* R*). Then*

(a) $f(z) + \phi(z)$ *is of non-analyticity* $\leqq r_1 + r_2$ *at* z_0 *(or in* R*)*

(b) $f(z)\phi(z)$ *is of non-analyticity* $\leqq |f(z_0)|\, r_2 + |\phi(z_0)|\, r_1$ *at* z_0

$$\left(\text{or} \leqq r_2 \max_{z \text{ in } R} |f(z)| + r_1 \max_{z \text{ in } R} |\phi(z)| \text{ in } R\right).$$

2. A generalization of Cauchy's fundamental theorem.

THEOREM 6. *If* $f(z)$ *is of non-analyticity* r *in a simply connected region* R *and if* C *is a rectifiable simple closed curve lying entirely within* R*, then*

$$\left|\int_C f(z)dz\right| \leqq 4\sqrt{2}\Omega r,$$

where Ω *is the area enclosed by* C. [4].

The proof of this theorem is a simple modification of the ordinary proof of Cauchy's fundamental theorem. It runs as follows.

Proof. Let us divide the whole plane into equal squares with sides of length d and parallel to the real and the imaginary axes respectively. Consequently the interior A of C is divided into a number of sub-regions. Let us re-divide each of the sub-regions in the manner of theorem 3. Then A is finally divided into a number of regions whose boundaries are squares $p_1, \cdots, p_M$ and regions whose boundaries $q_1, \cdots, q_N$ are portions of sides of squares and parts of C. Then

$$\int_C f(z)dz = \sum_{i=1}^{M}\int_{p_i} f(z)dz + \sum_{j=1}^{N}\int_{q_j} f(z)dz \tag{1}$$

where the paths of integration are taken counter-clockwise. According to theorem 3, we can find a point u_i within p_i and a point v_j within q_j such that

$$f(z) - f(u_i) = f'(u_i)(z - u_i) + \eta_i(r + \epsilon)(z - u_i), \qquad |\eta_i| \leqq 1, \quad \text{for } z \text{ on } p_i,$$

$$f(z) - f(v_j) = f'(v_j)(z - v_j) + \zeta_j(r + \epsilon)(z - v_j), \qquad |\zeta_j| \leqq 1, \quad \text{for } z \text{ on } q_j,$$

where ϵ is a given positive number.

Then

$$\int_{p_i} f(z)dz = f'(u_i)\int_{p_i}(z-u_i)dz + (r+\epsilon)\int_{p_i}\eta_i(z-u_i)dz$$

$$= (r+\epsilon)\int_{p_i}\eta_i(z-u_i)dz$$

and

$$\left|\int_{p_i} f(z)dz\right| \leqq 4\sqrt{2}\, c_i^2(r+\epsilon)$$

where c_i is the length of the sides of p_i. Similarly

$$\left|\int_{p_i} f(z)dz\right| \leqq (4d_j + l_j)\sqrt{2d_j}(r+\epsilon),$$

where l_j is the length of the curved part of q_j and d_j is the side-length of the square consisting a part of q_j. By (1),

$$\left|\int_C f(z)dz\right| \leqq 4\sqrt{2}\,(r+\epsilon)\left(\sum_{i=1}^{M} c_i^2 + \sum_{j=1}^{N} d_j^2 + ld\right),$$

where $l = \sum_{j=1}^{N} l_j$ is the length of C.

Let $d\to 0$, $\epsilon\to 0$, then

$$\sum_{i=1}^{M} c_i^2 + \sum_{j=1}^{N} d_j^2 \to \Omega$$

and the theorem follows.

3. The approximation of a non-analytic function by an analytic function. It is well-known that a real continuous function can be approximated by a real continuous function, as accurately as we please. A non-analytic function of a complex variable, however, cannot be approximated by an analytic function with arbitrary degree of accuracy. Before handling this problem, we have to generalize Cauchy's fundamental formula.

THEOREM 7. *If $f(z)$ is of non-analyticity r in a simply connected region containing the rectifiable simple closed curve C, we have for any z inside C,*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} + kr, \qquad k \leqq 4\sqrt{2}\left(1 + \frac{\Omega}{2\pi}\right),$$

where Ω is the area enclosed by C. [5].

Proof. 1) Suppose C lies within the unit circle about z. (A part or the whole of C may coincide with the circle.) Let us describe, within C, a small circle about z, and join c and C by a cut l so that the region S bounded by c, C and l is simply connected. Let Γ be the boundary. Then

$$\int_C \frac{f(w)-f(z)}{w-z}\,dw = \int_\Gamma - \int_c = I_1 - I_2,$$

say, where the paths of integration are taken counter-clockwise.

We may choose the radius δ of c so small that, on c,

$$f(w) - f(z) = (w - z)f'(z) + \eta(r + \epsilon)(w - z), \qquad |\eta| \leqq 1.$$

Then

$$I_2 = \int_c f'(z)dw + (r + \epsilon)\int_c \eta dw = (r + \epsilon)\int_c \eta dw,$$

$$|I_2| \leqq 2\pi\delta(r + \epsilon).$$

By theorem 5, the non-analyticity of $(f(w) - f(z))/(w - z)$ at any point within Γ is $\leqq r/|w - z|$, since the non-analyticity of $(w - z)^{-1}$ within Γ is 0.

Dividing the interior S of Γ into any number of sub-regions with boundaries $c_1, \cdots, c_N$, we have

$$I_1 = \sum_{i=1}^{N} \int_{c_i} \frac{f(w) - f(z)}{w - z}\, dz$$

where the paths of integration are taken counter-clockwise. By theorem 6,

$$|I_1| \leqq 4\sqrt{2} \sum_{i=1}^{N} \Omega_i \delta_i r$$

where Ω_i is the area enclosed by c_i and δ_i is the least upper bound of $1/|w - z|$ within and on c_i. By the definition of double integral,

$$|I_1| \leqq 4\sqrt{2}\, r \iint_S \frac{dxdy}{\sqrt{(x - a)^2 + (y - b)^2}} \leqq 4\sqrt{2}\, r \int_\delta^1 d\rho \int_0^{2\pi} d\phi \leqq 8\sqrt{2}\, \pi r$$

where (a, b) are the coordinates of z.

Making $\delta \to 0$, we have $\epsilon \to 0$ and

$$\left| \int_C \frac{f(w) - f(z)}{w - z}\, dw \right| \leqq 8\sqrt{2}\, \pi r.$$

2) Now consider the case that C does not lie entirely within the unit circle described about z. The arc of the circle, which lies within C, will divide the region enclosed by C into two parts. Let Γ_1 and Γ_2 be their boundaries. Let Γ_1 include z. By theorem 5, $(f(w) - f(z))/(w - z)$ is of non-analyticity $\leqq r$ inside and on Γ_2. By 1) and theorem 6, we have

$$\left| \int_C \frac{f(w) - f(z)}{w - z}\, dw \right| \leqq \left| \int_{\Gamma_1} \right| + \left| \int_{\Gamma_2} \right| \leqq 8\sqrt{2}\, \pi r + 4\sqrt{2}\, \Omega r.$$

The theorem follows immediately.

The theorem gives an analytic approximation of $f(z)$, since the function $\int_C f(w)dw/(w - z)$ is analytic within C. The error is kr which depends on the non-analyticity r of $f(z)$ and the area Ω of the domain of validity of the formula. From this theorem, we can deduce an approximate Taylor's expansion of $f(z)$ with the same error kr.

REFERENCES

This paper was received from the author without suitable references or indications of which portions of the paper are presentations of the work of other mathematicians and which portions are due to the author. Since it was not feasible to communicate with the author, it was decided to ask some American mathematician to supply suitable notes. This work was done by Professor John De Cicco of Illinois Institute of Technology.

THE EDITOR

1. A complex function $w=X(x, y)+iY(x, y)$ of the complex variable $z=x+iy$, where the dependent real functions (X, Y) possess continuous partial derivatives with respect to the real variables (x, y) but do not necessarily satisfy the Cauchy-Riemann equations, has been termed a *polygenic function* of z by Kasner. If the Cauchy-Riemann equations are satisfied, then the function is, of course, monogenic in z. See Kasner, A theory of polygenic (or non-monogenic) functions, Science, 66, 581–582, 1927.

2. This derivative circle has been called the *Kasner circle* by Hedrick and others. Kasner first obtained this circle in 1927 and has since developed a very interesting geometry of the derivative of a polygenic function. See Hedrick, Non-analytic functions of a complex variable, Bull. Amer. Math. Soc. 39, 75–96, 1933.

3. What the author defines as the derivative of a polygenic function is really the mean derivative. As an operator, the *mean derivative* is

$$\mathfrak{D} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and the *phase derivative* is

$$\mathfrak{P} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If $w=X+iY$ is an analytic polygenic function, that is, if X and Y are each Taylor's Series in (x, y), then the mean derivative is the formal partial derivative of w with respect to $z=x+iy$, and the phase derivative is the formal partial derivative of w with respect to $\bar{z}=x-iy$.

4. This result is intimately related to Pompieu's areolar derivative theorem which states that

$$\lim_{C\to 0} \int_C \frac{f(z)dz}{\sigma} = 2i\mathfrak{P}[f(z)]$$

where σ is the area enclosed by C. See D. Pompieu, Rendiconti di Palermo, Vol. 33, 108–113, 1912 and Vol. 35, 277, 1913.

5. This may be obtained as a consequence of Pompieu's representation of a polygenic function, which is the following

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} + \frac{1}{\pi}\iint_R \frac{\mathfrak{P}[f(w)]dxdy}{z-w}.$$

See also the following papers which contain a bibliography of other papers on polygenic functions.

Kasner, The second derivative of a polygenic function, Trans. Amer. Math. Soc. 30, 803–818, 1928.

Kasner and DeCicco, The derivative circular congruence representation of a polygenic function, Amer. Journal Math. 61, 995–1003, 1939.

Finally see a forthcoming paper by DeCicco, Survey of polygenic functions, Scripta Mathematica, 1945.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Kappa Mu Epsilon, Central Michigan College of Education

Meetings are held twice a month. Eighteen new members were initiated in the fall of 1943, and thirteen in the spring of 1944. Due to the Navy V-12 program, some members have been transferred to advanced training. By July 1 many more will have left. The membership at the present time totals thirty-three. Although the membership has been varying every four months due to transfer of the V-12 students to advanced training, there has not been a noticeable decline in the activities while they have been with us. The necessity of mathematics is known to all of these men who have left us.

Discussions at two meetings in March and April dealt primarily with the publication of a pamphlet. The main object of this activity is to discuss the mathematical requirements in various fields (science, business, teaching, engineering, *etc.*). Upon the completion of the pamphlet it will be distributed among high school students and it is expected to clarify and help them in the selection of mathematical courses while in high school.

The following talks, from one-half to one hour in length, were presented during the winter and spring.

The Mercator chart, by Paul Decess

Mathematical recreations, by Fran Teel

Cryptanalysis, by Dr. Cleon Richtmeyer

History of calculus and its application to physics, by Bob Murch.

The annual picnic was held early in June. Officers for the fall term of 1943 were: President, Paul Brown; Vice-President, Emma Skinner; Secretary, Betty Sack; Treasurer, Charles Turner. Officers for the spring term of 1944 were: President, Emma Skinner; Vice-President, George Yerganian; Secretary, Betty Sack; Treasurer, Joyce Sherwood. Officers for next fall are: President, Joyce Sherwood; Vice-President, Pauline Nelson; Secretary, Martha Toles; Treasurer, Margaret Beck; Corresponding Secretary, L. C. Gray.

Delta-x, University of the City of Toledo

At the first meeting of *Delta-x* for 1943–44, the members of the club amended a requirement that all members must be studying or have studied calculus, so that students taking college algebra, trigonometry or analytic geometry are eligible for associate membership. This action has resulted in an active membership nearly as large as that of other years despite the large number of members

called to the armed forces. Papers of interest to students of mathematics were presented at each of the regular monthly meetings:

Series, by Tom Jarrett
Geometrics of paper folding, by Margaret Kauffmann
Special properties of the conics, by Ellsworth Shinavar
Computation of pi, by Larry Muttart
Computation of e, by Lois Martin
Curve fitting, by William Bergman
Sun dials, by William Lunn
Geometric proofs of trigonometric formulas, by Mildred Schalkhauser

The annual *Get-Acquainted Roast* opened social activities of the club. The *Hard Times Party* was held in November and a Christmas party was held after the regular December meeting. In January the members enjoyed an ice skating party. The Annual *Delta-x* Banquet was held in May. The speaker was Dean John Brandeberry, a faculty member of the club. His topic was:

Imaginary branches of real curves.

The last function of the club for the year was a picnic. *Delta-x* yearbooks were presented to all members. The co-editors were Margaret Kauffmann and Lois Martin.

Officers elected for the coming year were installed after the banquet meeting by the retiring officers and Professor Dancer, Faculty Adviser. They are: President, William Lunn; Vice-President, Lois Martin; Secretary-Treasurer, Margaret Kelley.

Mathematics Club, Montana State University

The following talks were presented during 1943–44:

The life and work of Fermat, by Phyllis Biddle
Mathematician's apology, by Grayce Miller
Mathematics in big game hunting, by Dr. Roy Dubisch
Anecdotes on mathematicians, by Dr. Harold Chatland
Take-off on mathematics, by Mrs. Mel-Iden Chambers
Logarithms, by Kenneth Axvig
Navigation, by Dr. Harold Chatland
Mathematical tricks, by Mark Jacobsen
Amusements in mathematics, by Anna Harwick
Mathematical nuts, by Elizabeth Nelson
The basis of geometry, by Dr. N. J. Lennes.

The annual party for members of the Mathematics Club and their guests was held at the home of Dr. Chatland on February 18th. After Miss Nelson's talk on May 4th, the Club, together with the university chapter of *Pi Mu Epsilon* decided to have pictures taken of our two retired professors of mathematics, Dr. N. J. Lennes and Dr. E. F. A. Carey, to be framed and hung in the Mathematics Library of Craig Hall. The next afternoon the annual Math-Chem Clubs picnic was held at Greenough Park in Missoula. Officers of the club were: President, Phyllis Biddle; Secretary, Anna Harwick; Treasurer, Grayce Miller.

Mathematics Club and Kappa Mu Epsilon, College of Wooster

Kappa Mu Epsilon held only two meetings during the year, because of the limited number of advanced mathematics students. At the last meeting five new members were elected and it is hoped that there will be enough members to warrant regular meetings next year.

The *Mathematics Club* has met regularly every two weeks during the college year and considered the subjects of *Navigation and Meteorology* as they pertain to flying. Professor C. O. Williamson, who is a licensed pilot and has taught navigation in the Navy Preflight School, did most of the lecturing and illustrated with interesting experiences. A picnic was held in September and a Christmas party in December. The officers elected for 1944–45 are: President, Anne Widener; Vice-President, Janet Thompson; Secretary-Treasurer, Lois Hayenga.

Junior Mathematical Club, University of Chicago

During the year 1943–44 the *Junior Mathematical Club* of the University of Chicago continued its fortnightly meetings and teas, and supplemented this schedule with a party in the Winter Quarter and a picnic in the Spring Quarter. At the regular meetings, the following papers were presented to the Club:

Topological spaces, by Robert Kates

Mathematics in dynamic meteorology, by Lawrence Markus of the Institute of Meteorology

Mathematical paradoxes, by Professor E. P. Northrup

Functional equations, by Daniel Zelinsky

A mathematical problem in traffic control, by Professor L. R. Wilcox, Illinois Institute of Technology

Fourier coefficients, by Marie Wurster

Soap films and the calculus of variations, by Professor W. T. Reid

Dynamics of rigid bodies, by Daniel Zelinsky

Some problems in contour integration, by Sidney Dancoff of the Metallurgical Laboratory

Product complexes, by Hyman Zimmerberg

Relative motion and the accelerations of Coriolis, by Professor E. P. Lane

A matrix method in curvilinear coordinates, by George Platzman of the Institute of Meteorology

Transformations of surfaces, by Mary Petty.

The student paper adjudged to have the best mathematical content and presentation was that on *Dynamics of rigid bodies*: the prize awarded to Mr. Zelinsky for this paper was a copy of Weyl's *Classical Groups*.

Officers of the Club for 1943–44 were: President, Daniel Zelinsky, Social Chairman, Mary Petty; Treasurer, Hyman Zimmerberg; Committee, Charles Nichols and Marie Wurster. Officers-elect for 1944–45 are: President, Charles Nichols; Social Chairman, Marie Wurster; Treasurer, Betty Alexander; Committee, Verna LaMantia and Herman Rubin.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry. By K. B. Patterson and A. O. Hickson. New York, F. S. Crofts and Company, 1944. 10+187 pages. $2.10.

To quote from the preface, "this book contains topics usually treated in a brief course on analytic geometry and can be discussed thoroughly in from forty to fifty class periods. . . . it contains many examples and more figures than are usual in a brief course in the subject." The format is pleasing and the figures are clear though somewhat small. A list of the chapter headings should indicate well enough the general contents of the book: 1. Cartesian Coordinates. 2. The Loci of Equations. 3. The Equation of a Locus. 4. The Straight Line. 5. The Circle. 6. Conic Sections. 7. Translation and Rotation of Axes. 8. Polar Coordinates. 9. Parametric Equations. 10. Tangents and Normals. 11. Analytic Geometry of Three Dimensions. The problems are of the type usually found in elementary American textbooks and seem adequate for the purpose. In chapter 10 the notions of limit and derivative are introduced and applied to the theory of tangents.

Although the book makes no attempt to give a true axiomatic approach to analytic geometry, it illustrates a recent tendency in mathematical texts which deserves encouragement. (It seems to the reviewer that the titles of chapters 2 and 3 are very revealing as to the spirit of the authors.) Naturally the book assumes a knowledge of the elements of algebra and trigonometry, but the authors have tried to supply carefully explicit definitions at appropriate points in the development of the subject. As the following examples should indicate, theorems are explicitly stated, usually with great care: "Theorem 16. The equation of every straight line is of the first degree." "Theorem 17. Every equation of the first degree in x and y represents a straight line."

It would appear that this book could be materially improved by the addition of perhaps five pages, in the form of qualifying sentences, additional remarks and definitions, and a few expanded or corrected proofs. The reviewer finds himself in disagreement with about forty items scattered throughout the text, most of which represent comparatively trivial slips easy of correction. (We are not speaking of typographical errors; of the latter we found only two, both perfectly obvious.) As a typical example we note on page 155 the italicized false statement that "the slope of the parabola $y^2=4ax$ at any point (x, y) on the parabola is $dy/dx=2a/y$." Despite the fact that the authors are careful at more than one point to ban division by zero, a number of theorems and proofs are false because

of the neglected possibility of zero denominators. As a further example, Theorem 24 (page 56) gives a false "necessary and sufficient condition" (in determinantal form) that three straight lines should have a common point. The criterion fails in the case of three parallel lines, and the "proof" moreover tacitly assumes (in the absence of any explanatory remark) that no two of the lines are parallel. An equally serious matter (page 15) is a "description," not formally presented as a definition, of an algebraic equation; this description would embrace among others the equation $y=\sin x$. Again, on pages 34–35 the authors raise the important question as to whether the graph of an equation derived from a geometrically defined locus will necessarily represent all of that locus and nothing else; but the example they give appears to be nicely calculated to quiet any doubts which a student might entertain. It seems unfortunate moreover that this remark should be recalled several times in connection with the standard equations of the conic sections.

Lest there be any suggestion of bias in this review we hasten to mention an excellent section on horizontal and vertical asymptotes, a well-phrased discussion of families of lines, and a good chapter on transformation of axes. At the end of the chapter on conics there are in addition some geometrical proofs of the fact that conics may be obtained as plane sections of a cone. (The last of these proofs seems to contain an obvious slip.)

If a revised edition of this book should be brought out it could easily be made into a first rate example of its kind.

R. H. BRUCK

Tables of Lagrangian Interpolation Coefficients. Prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency, conducted under the sponsorship of the National Bureau of Standards. New York, Columbia University Press, 1944. 36+392 pages. $5.00.

TABLE I (extent 100 pages) gives three-point interpolation coefficients $p(p+1)$, $(p-1)(p+1)$ and $p(p-1)$ to 9, 8 and 9 decimals from $p=0.0000$ to $p=1.0000$. TABLES II to IX give four-point to eleven-point coefficients. Extent of TABLE IX is 2 pages (10 decimals). In the reviewer's opinion, 2 pages for TABLE I and 100 pages for TABLE IX would offer more. TABLE X gives interpolation coefficients from 3 to 8 points for p at intervals of 0.1. TABLE XI does the same for p in multiples of 1/12. TABLES XIIa and XIIb give Lagrangian integration polynomials and integration coefficients. A 22-page introduction by A. N. Lowan gives a concise and thorough exposition of Lagrangian interpolants with an analysis of the error and applications to numerical integration. This publication definitely is an excellent help for the specific purpose of interpolation and numerical integration. The cover size is 11″ by 8″, the photo-offset reproduction is immaculately distinct and clear.

M. A. SADOWSKY

Basic Air Navigation. By E. E. Blackburn. New York, McGraw-Hill Book Company, Inc., 1944. 7+296 pages. $3.00.

Here is a text in which are presented in orderly manner all the problems that the air navigator must deal with in planning a flight; in flying his airplane at the proper height, with the right speed, and in the correct direction; and, finally, in bringing his craft safely to its destination. The author states in the Preface that the book is written to assist the pilot in his rigorous training in navigation, and that he has endeavored to keep the subject simple. The author's years of experience in flying and in teaching navigation, and his ability to express in simple, understandable language the various phases of the problems confronting the navigator prove that he has successfully accomplished the task he set for himself.

The few examples involving numerical calculations require the ability to do the simple arithmetic operations. The triangle of velocities is solved graphically or by means of a computer. There is one reference (page 179) to a sine, but no knowledge of the trigonometric functions is required, for tables are provided giving necessary data where such values are needed. The slide rule (especially, the circular slide rule) and the ground speed computer are recommended for use where possible. Charts, maps, and graphs are used freely in solving problems involving direction, distance, gas load, gas consumption, altitude, speed, time, *etc.*

Problems of flying by day and by night, over land and water, with and without land or water marks, with radio aids, and with celestial bodies as guides, are all treated carefully. Methods of determining lines of position and fixes are explained. Two chapters are devoted to the principles and practices of celestial navigation.

The tools of the navigator are pictured and their uses are explained in connection with the subject as it progresses. These tools include sample pages from the tables H. O. 208 and H. O. 214; a list of fifty-five navigational stars, and diagrams for locating them; sample pages from the *American Air Almanac* giving appropriate data on the solar-system bodies, sun, moon, Venus, Mars, and Jupiter; and such instruments as the computer, altimeter, compass, air-speed indicator, drift indicator, drift bombs, octant, and chronometer.

In the final chapter the author gives general instructions to the student navigator. He emphasizes the points to be considered in preparing for the flight, in the course of the flight, and upon approach of the terminal point of the flight. He advises careful planning of the flight and alertness at all times.

The text contains a few errors and defects of which the most noticeable are: (1) The last part of the first sentence on page 137 " · · · , he will either add or subtract 90° from this direction" should read something like this, " · · · , he will either add 90° to or subtract 90° from this direction." (2) Commas should follow the words "altitude" and "temperature" on page 190, third line from the bottom of the page. (3) The sentence beginning on page 260, fourth line from the

bottom, is misleading. The sentence should state clearly that when the sun, planets, and stars are viewed from two points, one on the earth's surface and the other at the earth's center, the rays of light from any one of these bodies may be considered parallel. (4) On page 203, eighth line from the bottom, the distance of the closest star is mentioned as "so many million light-years." This, of course, is an exaggeration, for the distance of the nearest star is slightly more than four light-years. (5) The statements in the last paragraph, page 217, concerning the earth's rate of travel in miles per minute and the apparent motion of the sun are not accurate. (6) In the lower part of FIG. 253, page 215, the star shown as Al N'air is correctly spelled Al Na'ir. (7) The two stars in FIGS. 241 and 248 designated by the Greek letter δ (delta) should be ϵ (epsilon). (8) The footnote, page 247, contains the statement that "FIGURE 275 is also contained in the pocket in the back of the book." The list of Chart Problems on the pocket did not include this as one of the figures in the pocket and it was not found there. It is given, however, on page 248 of the text. (9) In FIGS. 291a and 298, and in the text below FIG. 291a the bubble knob is designated by a large "B," while in the last paragraph on page 265 the knob is referred to by a small "b." (10) The data on the extreme right margin of FIG. 320 are incomplete, probably due to faulty printing.

The errors and defects listed above should not be weighed too heavily against the value of the book, for the reviewer believes, after a careful examination of this text, that the author and publishers of *Basic Air Navigation* have combined in giving to the student navigator a text of very high class, and in no case will any of the faults discovered lead the pilot to any error in the preparation for, or in the execution of, the flight. The text should prove valuable to the student navigator.

H. H. DOWNING

Navigation. By L. M. Kells, W. F. Kern, and J. R. Bland. New York & London, McGraw-Hill Book Company, Inc., 1943. 20+479 pages. $3.75.

This textbook on navigation is a complete exposition of the information necessary to determine one's location on the surface of the earth and of the "rules of the road" in going from place to place. The chief emphasis is laid on detailed procedure in making measurements, recording the data, manipulating the numerical values and interpreting the numerical results. Mathematics, as such, is distinctly avoided and the more difficult portions are relegated to an appendix which may be consulted by those students who have the curiosity and the desire to justify the methods.

The first chapter explains most of the mathematical terms and operations and it shows the student how to insert values in a formula, to use a slide rule or a table of logarithms. The rest of the book is devoted to a description and use of the various instruments, to charts, almanacs, signals, navigation laws and rules, range finding, the elements of astronomy and star identification. The basic formulas of spherical trigonometry are given in a ten-page chapter.

The various methods of piloting are carefully outlined. These include the astronomical methods, dead reckoning, sighting on lighthouses, buoys and landmarks and radio beams. A large selection of exercises and answers are very helpful for both the student and the instructor.

There are many excellent line diagrams. In addition, the book is profusely illustrated with inspirational pictures, principally official Navy photographs. Most of these have explanatory notes below them although a number of them have no necessary connection with the adjacent text. This book is definitely intended primarily for Navy personnel. It emphasizes Navy methods, charts and ship and plane maneuvers.

The thoughtful arrangement of the text and sketches, the use of heavy type for emphasis and the high quality of the typography makes this book very readable.

MICHAEL GOLDBERG

Basic Marine Navigation. By B. J. Bok and F. W. Wright. Boston, Houghton Mifflin Co., 1944. 8+422 pages. $4.50.

The major portion of this text is devoted to geo-navigation, with special emphasis on coastwise navigation. Piloting, Dead Reckoning, and the compass are treated thoroughly, but the sailings are passed over lightly. The only chart that receives much attention is the Mercator projection, and here the development is intuitive. The chapter on tides and currents and the chapter on meteorology make interesting and instructive reading. In the treatment of celestial navigation, the authors have tried to use only terms that are essential. Nothing is said of the real motion of the planets, the whole treatment is from the viewpoint of apparent motion. The chapter on the sun is well done, but the problem of finding a fix from the stars, planets, and moon is rather hastily passed over. The solution of the astronomical triangle is mainly made through the use of the H. O. 214 tables. However, a short chapter is devoted to the discussion of H. O. 208 and H. O. 211. The last chapter of the book is devoted to the problem of navigating a lifeboat.

The print is good, and the authors have an interesting style. There are, however, very few proofs as the reader is not assumed to have a knowledge of trigonometry. The general plan of attack throughout the book is to state the problem and how to solve it, followed by a well chosen illustrative example.

A *Kit* containing maps and practice material is published separately from the text, but is needed for the solution of some of the problems of the book. While the text was drafted by the first-named author, Bok, and the problems by Wright, the problems correlate nicely with the material of the book and are well stated and instructive.

The authors state that "the book and the *Kit* together provide a complete outfit for a self-taught course in navigation." However, many teachers will feel that more proofs and more problems are needed when the book is used as a textbook for a course in navigation.

F. G. DRESSEL

Mathematics for Exterior Ballistics. By G. A. Bliss. New York, John Wiley and Sons, Inc., 1944. 7+128 pages. $2.00.

This small book is based upon material used several times in courses given by the author at the University of Chicago. Professor Bliss, widely known and respected for his research, teaching and direction of graduate training in mathematics, was one of the small but enthusiastic band of research mathematicians in the Range Firing Section at Aberdeen Proving Ground during part of World War I. Professor Bliss contributed notably to the handling of differential corrections by applying the notion of functions of lines and introducing into practical ballistic computations appropriate adjoint equations.

The book is written in a clear and interesting style with material suitable for undergraduate study. After an opening chapter largely concerned with military maps, separate chapters treat of: II. The differential equations of a trajectory, III. The Siacci theory, IV. The approximate integration of the equations of exterior ballistics, V. Differential corrections, VI. Bombing from airplanes. A supplement consists of Tables for Computations. There are a few short sets of exercises headed "Examples" scattered through the book.

There seems to be no question that this is one of the most attractive brief American texts on this now timely subject.

One mild comment has been made by several local readers. It does not affect the suitability of the text for preliminary study:

To anyone currently engaged in ballistic computation under circumstances analogous to those to which the author of this text was introduced during World War I, there comes a curious sense of suspended animation on reading in this text the numerous unqualified assertions purportedly describing present practice. The reader, an unwilling Rip van Winkle, is inclined to turn to the cover and gaze incredulously at the printed date "1944." The terms and the practices here reported often sound unrealistically quaint to the present staff of young computers. They never use the term "range table"; the "standard trajectory" of today is not that described in the text. The corrections for sphericity are not now made as here described, and the corrections for the rotation of the earth are at present handled in a much simpler manner. Other routine corrections, much more important than these, are missing completely in this text. Even in such a static matter as map reading this text is out of date. The recent basic field manuals on map reading have freed themselves from some of the French overlay of terminology whose only excuse for original adoption lay in the special adjustment problems incident to American participation with an actively engaged army of France. More specific and detailed data might be reasonably expected from the reviewer to support these unenthusiastic comments, except for the fact cited by the author himself in the preface: "Not much can be said here . . . . They are closely guarded secrets, naturally not available to a civilian writer."

A. A. BENNETT, Major, Ord. Chief, Computing Branch
Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

Celestial Navigation. A Problem Manual. By Walter Hadel. New York and London, McGraw-Hill Book Company, 1944. 13+261 pages. $2.50.

"This collection of practical problems in celestial navigation, including various calculations in simple dead-reckoning navigation, has been put together in the form of a text, or guide, to a course in celestial navigation." (From the preface.)

In the many texts on navigation which have appeared lately and are still appearing frequently, each author has his own ideas about what should be included and what a text-book should contain. Some go to the extreme of simplicity and others give in great detail every phase of the subject. This author goes on the assumption that the student should learn by practice. He believes that the student should be able to work the necessary problems with accuracy first and speed second in importance. He bases his course on an orderly succession of "Celestial Quizzes" which covers the necessary material with a minimum of theory and a maximum of practice.

There are 46 Celestial Quizzes, including four in Appendix A, arranged as lessons would be in an ordinary text. Each quiz begins with a page (more or less) of explanatory notes, then states the problems to be solved and solves a few, leaving the remainder to be worked out by the student.

For example, the first quiz begins with the averaging of time and angle. This must be done in reducing all observations with the sextant. Next is immediately introduced the standard practice of determining LHA (Local Hour Angle) and changing from Local Standard Time to Greenwich Civil Time, and the first lesson ends with the method of drawing azimuths. It seems likely that a student would have some difficulty in beginning his study of navigation from this book by himself and that he would need some help from an instructor.

In the succeeding lessons or quizzes, the author takes up such subjects as charts, lines of position, running fixes, corrections to observations and compass corrections by the method of short explanations and problems, some worked out and some left for the student.

One desirable feature of the book is the establishment of time limits for working the problems and a speeding-up of work as the student becomes more proficient. The whole idea is attractive to the instructor who wants a series of exercises for his students either for use with an explanatory text or with classroom lectures on the theory of navigation. The mathematics is reduced to addition and subtraction, although the development of the formulas for H. O. 211 is included in the Appendix. This development is based on spherical trigonometry, the details of which are left to the instructor.

All the work is based on modern methods, using the American Air Almanac and H. O. 214 and H. O. 211. Sufficient extracts are given in the Appendix B to enable the student to work all problems. Also there are a few examples from Weems' Star Altitude Curves—a rapid but somewhat inaccurate system of obtaining pre-computed altitudes.

It is obvious that the book has been prepared from the viewpoint of the air navigator. However, the air methods are being adopted, in part at least, by mariners, even to the extent of using the Air Almanac.

For a problem manual supplementing a course in the theory of navigation or where the practice is given without the theory, this book should be very useful and is recommended for careful consideration by all instructors in the field of navigation.

C. M. HUFFER

Riddles in Mathematics. A Book of Paradoxes. By E. P. Northrop. New York, D. Van Nostrand Company, 1944. 8+262 pages. $3.00.

This volume comprises a pleasantly informal collection of illustrative examples and problems illustrating common errors, interspersed with comments of historical and philosophical nature. Topics range from divison by zero to Russell's theory of types. The material might furnish entertaining supplementary reading for high school and undergraduate mathematics students, and persons with slight knowledge of the subject could enjoy at least the grammatical and topological "paradoxes."

After the two introductory chapters with paradoxes "for everybody," comes a chapter on arithmetic including the growth of powers of 2, Fermat numbers, division of the circle, and the game of Nim. Chapter IV, "paradoxes in geometry," starts with simple optical illusions and brings up among other things the Fibonacci series, curves of constant breadth, the Konigsberg bridge problem, Klein's bottle, and the four color problem. Chapters V and VI on "algebraic fallacies" and "geometric fallacies" demonstrate incorrect proofs of things such as $-1=+1$ and any triangle is isosceles. Chapter VII, "paradoxes of the infinite," brings in a bit of cardinal number theory, conditionally convergent series, and pathological curves. In Chapter VIII, on probability, some right, wrong, and uncertain methods of computing probability are included with a discussion of the theory and applications. Chapter IX on "paradoxes in logic" presents some of the difficulties in the foundations of mathematics, describing briefly the three schools of thought and discussing paradoxes like those of Richard and Burali-Forti. Chapter X on "paradoxes in higher mathematics" contains a brief and odd assortment of examples involving more formal mathematics, the most interesting being a demonstration of how in the computation of the surface area of a cylinder as the limit of inscribed polyhedra one may obtain different results.

An appendix gives brief explanations or discussions for problems posed by the author in the body of the book. A chapter by chapter bibliography and an index end the volume.

BERNARD VINOGRADE

Calculus Refresher for Technical Men. By A. A. Klaf. New York, McGraw-Hill Book Company, Inc., 1944. 8+431 pages. $3.00.

This book represents the extreme in the method of teaching by rote and example. As its title indicates, it is intended as a review of calculus rather than as a first presentation. In unique fashion it presents the subject catechistically, posing questions and following them with answers, which consist of brief unjustified statements usually followed by illustrations.

Most of the customary topics of the standard calculus text are covered, and numerous examples from diverse engineering fields are given. A notable feature is the complete detail with which the illustrative problems are solved, even to minute numerical calculations. A substantial portion of the book is devoted to formal processes, some sixty pages being spent, for example, on formal integration. The entire subject of series is omitted, and differential equations are barely introduced.

There are numerous mathematical errors and rhetorical peculiarities which cannot be overlooked because of their cumulative effect. For example, the term *function* is used to denote an equation connecting variables, rather than what is usually meant; no hint is given that functions are not always defined by analytic formulas. Discontinuous functions are those which become infinite, according to the author; certainly even technical men should know about finite discontinuities. Differentials are used before they are defined. Integrals are defined as antiderivatives, thus rendering a logical presentation of the fundamental theorem of integral calculus impossible. Indeed, the statement of this theorem is nothing short of nonsense. The author's motive for differentiating rational functions by first decomposing into partial fractions and then claiming the method superior must be obscure. In sequences of mathematical statements or equations the author omits punctuation, even final periods, apparently unaware that even mathematical discourse consists of sentences.

But more serious than such errors is the complete lack of exact ideas, careful definitions and proofs. The inevitable "explanations by example" must become extremely monotonous and irritating to any but the most stupid of students. The mathematician who sees this book cannot help feeling that his subject is being systematically robbed of its content and left only with the dregs of formal manipulations—all of which can benefit no one, least of all technical men, whose need for more not less, stress on fundamental concepts is admittedly desperate.

The only apparent legitimate use of the book is as a reference source for a few lesser known applications of calculus.

L. R. WILCOX

NEW BOOKS RECEIVED

Handbook of Air Navigation. By W. J. Vanderkloot. New York and London, McGraw-Hill Book Company, 1944. 13+333 pages. $3.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 641. *Proposed by F. M. Garnett, Savannah, Ga.*

A lighter, twenty by thirty feet, travels upstream at a uniform speed of 2 m.p.h., against a current of $1\frac{1}{2}$ m.p.h. A swimmer, whose rate of swimming is 4 m.p.h. in still water, swims around the lighter. How long does he take to complete the circuit?

E 642. *Proposed by V. Thébault, San Sebastián, Spain*

In the duodenary scale, a certain number does not end in 0, but its square does. Show that the square ends in 30, and the cube in 60.

E 643. *Proposed by W. E. Buker, Pittsburgh Public Schools*

The sides a, b, c, d of a plane quadrangle being given in order and also the area A, find the length of the longer diagonal.

E 644. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Do there exist determinants of order $n>2$, such that all the $n!$ terms of the expansion are positive?

E 645. *Proposed by C. D. Olds, Purdue University*

Let P_1, P_2, P_3 be any three points on a plane curve C, and O a point in the same plane. If the areas of the three triangles OP_2P_3, O_3PP_1, OP_1P_2 are connected by a relation independent of the coordinates of P_1, P_2, P_3, prove that C is a central conic and O its center.

SOLUTIONS

A Naval Encounter

E 606 [1944, 88]. *Proposed by H. T. R. Aude, Colgate University*

An enemy ship A is proceeding on a course directly east at its top speed of 25 miles per hour when it is spied by a more powerful ship B of a speed of only 20 miles per hour. At the instant of discovery the ship B is 10 miles directly south of A. On account of the shore line the enemy ship A must continue its course, while the course of B is altered to $N\theta E$ to bring A within range. If the effective range is not more than 7 miles, find the course of B that will bring

about the least distance AB, and the interval of time (while following this course) during which A will be within range.

Solution by Frank Hawthorne, Allegheny College. Consider the motion of B relative to A. Construct a vector of length 25 (miles per hour) west from B to G. With G as center strike an arc of radius 20, and draw the tangent from B to this arc, the point of tangency being C. Then GC is in the direction B should sail. Triangle BGC is the common 3, 4, 5 right triangle; hence

$$\theta = \arctan \tfrac{4}{3}.$$

Thus the motion of B relative to A is 15 miles per hour in the direction BC.

Construct a vector of length 10 (miles) north from B to A. With A as center strike an arc of radius 7 to cut BC in the points D and E. Let F be the midpoint of DE. Now, since triangles AFB and BCG are similar, we have $AF/AB = BC/BG = 3/5$. But $AB = 10$; so $AF = 6$. Also $AD = 7$; hence $FD = \sqrt{13}$ and $DE = 2\sqrt{13}$. Thus the distance within range is $2\sqrt{13}$ miles, and the time within range is $2\sqrt{13}/15$ hours or $8\sqrt{13} = 28.85$ minutes.

Also solved by E. F. Allen, Murray Barbour, Colin Blyth, W. B. Campbell, L. R. Chase, Howard Eves, A. E. Gault, H. M. Gehman, R. W. Hamming, L. M. Kelly, D. A. Oliver, Jr. (using a real "maneuvering board"), E. K. Paxton, E. P. Starke, C. W. Topp, and W. Unterberg.

Barbour remarks that it is unreasonable to expect B to continue on the same course until A is out of range. By changing to a more easterly course as soon as A is within range, B could increase the time within range to 39.5 minutes.

A Number Divisible by the Sum of its Digits

E 608 [1944, 88]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

A certain three-digit number yields a quotient of 26 when divided by the sum of its digits. If the digits are reversed, the quotient is 48. What is the smallest three-digit number for which this is possible?

Solution by Irving Kaplansky, New York, N. Y. We make use of only the *first* of the conditions. By casting out nines, the sum of the digits is a multiple of nine, *viz.*, 9, 18, or 27. Thus the smallest such number is $9 \cdot 26 = 234$ (and the only other is $18 \cdot 26 = 468$, as $27 \cdot 26$ has four digits).

Also solved by F. A. Alfieri, E. F. Allen, Murray Barbour, C. B. Barker, Colin Blyth, W. H. Bradford, D. H. Browne, W. E. Buker, H. N. Carleton, R. E. Crane, Marian E. Daniells, Monte Dernham, William Douglas, Howard Eves, A. E. Gault, R. W. Hamming, Frank Hawthorne, T. W. Jackson, Aida Kalish, R. J. Koch, Helen McDevitt, Walter Penney, P. W. A. Raine, W. C. Rufus, E. D. Schell, M. J. Sheehy, E. P. Starke, P. D. Thomas, W. Unterberg, Jeanette Van Os, Alan Wayne, Michael Wilensky, Hazel Schoonmaker Wilson, W. R. Talbot and the proposer.

Starke remarks that the problem has the same solution if we interpret the

hypothesis as requiring divisions where there is a quotient and a remainder, the quotient only being considered.

Diagonals of Faces of a Parallelepiped

E 609 [1944, 88]. *Proposed by Frank Hawthorne, Allegheny College*

Show that the diagonals of three faces of a parallelepiped, drawn from the same vertex and prolonged half their length, determine three points which are coplanar with the opposite vertex.

Solution by R. C. Buck, Cambridge, Mass. With the selected vertex as origin, take vectors a, b, c along the edges of the parallelepiped. The extended diagonals are then

$$3(b + c)/2, \qquad 3(c + a)/2, \qquad 3(a + b)/2.$$

The mean of these three vectors is $a+b+c$, which leads to the opposite vertex. Thus, not only is the opposite vertex coplanar with the three constructed points, but it is their centroid.

Also solved by E. F. Allen, W. H. Bradford, D. H. Browne, Howard Eves, A. E. Gault, R. W. Hamming, Irving Kaplansky, E. P. Starke, P. D. Thomas, J. A. Zilber, and the proposer.

Kaplansky remarks that the following generalization can be proved similarly. The diagonals of n cells of an n-dimensional parallelotope, drawn from the same vertex and prolonged by $1/(n-1)$, determine n points whose centroid is the opposite vertex.

Curves of Constant Width

E 610 [1944, 88]. *Proposed by Howard Eves, Syracuse University*

(*a*) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd.

(*b*) What is the least area that a closed curve of constant diameter d may have?

(Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction. See, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, 1938, p. 51.)

Partial Solution by the Proposer. By infinitesimal geometry we can very readily show that if r_1 and r_2 are the radii of curvature at a pair of opposite points on the closed curve, then $r_1+r_2=d$.

Let the intrinsic equation of the curve be $s=f(\phi)$, where $f(0)=0$. Then, if p is the perimeter of the curve, $f(2\pi)=p$. Since $f'(\phi)=r$, the radius of curvature, we have

$$f'(\phi + \pi) + f'(\phi) = d.$$

Integrating with respect to ϕ, we get

$$f(\phi + \pi) + f(\phi) = \phi d + c.$$

To determine the constant of integration c, we observe that when $\phi=0, f(\phi)=0$. Hence $c=f(\pi)$, and we have

$$f(\phi+\pi)+f(\phi)=\phi d+f(\pi).$$

Putting $\phi=\pi$ we get $p=f(2\pi)=\pi d$; which is part (a).

Part (b) seems to bear some similarity to the minimal problems of Besicovitch and Kakeya. See W. W. R. Ball, *Mathematical Recreations and Essays* (11th edition), p. 99.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4136. *Proposed by H. S. M. Coxeter, University of Toronto*

The equation

$$\cos \pi x+\cos \pi y+\cos \pi z=0, \qquad 0 \leqq x \leqq \tfrac{1}{2} \leqq y \leqq z \leqq 1,$$

has the trivial solutions $y=\frac{1}{2}$, $z=1-x$, and $y=2/3-x$, $z=2/3+x$. It has also the non-trivial solution $x=1/5$, $y=3/5$, $z=2/3$. Prove that it has no other rational solutions.

4137. *Proposed by P. Erdös, Purdue University*

Given an integer $x \leqq n^2/4$ which has no prime factor greater than n, show that $n! \equiv 0 \bmod x$.

4138. *Proposed by G. Pólya, Stanford University*

Given two positive integers p and q define

$$S_{p,q}=\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2p}-\frac{1}{3}-\frac{1}{5}-\cdots-\frac{1}{2q+1}+\frac{1}{2p+2}+\cdots$$
$$+\frac{1}{4p}-\frac{1}{2q+3}-\cdots-\frac{1}{4q+1}+\cdots,$$

$$P_{p,q}=\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots$$
$$\cdot\left(1-\frac{1}{2q+1}\right)\left(1+\frac{1}{2p+2}\right)\cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \cdots = S_{1,1} = 1 - \log 2,$$

and the product by rearranging correspondingly

$$(1 + \tfrac{1}{2})(1 - \tfrac{1}{3})(1 + \tfrac{1}{4})(1 - \tfrac{1}{5}) \cdots = P_{1,1}.$$

Show directly that

$$P_{p,q} = (p/q)^{1/2}$$

and hence derive the well known result

$$S_{p,q} - S_{1,1} = \log (p/q)^{1/2}.$$

4139. *Proposed by F. L. Miksa, Aurora, Ill.*

The distinct pairs of n objects, n odd, are arranged in n columns so that each column contains $(n-1)/2$ pairs and no one object occurs more than once in the same column, without regard to the order of the columns or of pairs in a column. Determine the number of different ways in which this can be done.

4140. *Proposed by V. Thébault, San Sebastián, Spain*

Find an integer of two digits such that the difference between its cube and that of the integer whose digits are those of the first in reversed order is a perfect square.

SOLUTIONS

Generalization of the Orthopole

4081 [1943, 264]. *Proposed by Otto Dunkel, Washington Univ.*

Through the vertices of the triangle ABC parallels $A\alpha$, $B\beta$, $C\gamma$ of arbitrary direction are drawn meeting the transversal Δ in the points α, β, γ; and through the latter points straight lines are drawn parallel respectively to BC, CA, AB rotated through the angle θ, thus forming a triangle $A_1B_1C_1$ similar to ABC. Prove that: (1) As the direction of the parallels varies the vertices A_1, B_1, C_1 describe straight lines concurrent in a point $\phi(\theta)$. (2) For the particular set of parallels $A\alpha$, $B\beta$, $C\gamma$ which have the direction of Δ rotated through the angle $-\theta$, the triangle $A_1B_1C_1$ reduces to the point $\phi(\theta)$. (3) The locus of $\phi(\theta)$ is a unicursal cubic passing through the circular points at infinity, the point at infinity of the Newton line (ABC, Δ), the orthopole of Δ with respect to ABC, and A_0, B_0, C_0, the points of intersection of the sides of ABC with Δ.

Solution by the Proposer. Two sets of parallels through A, B, C meet Δ in the two sets of points $\alpha_i, \beta_i, \gamma_i$; $\alpha_j, \beta_j, \gamma_j$. From the similar triangles β_iA_0B, γ_iA_0C we have $A_0\beta_i/A_0\gamma_i = A_0B/A_0C = A_0\beta_j/A_0\gamma_j$ and $A_0\beta_i/A_0\beta_j = A_0\gamma_i/A_0\gamma_j$. The second set of straight lines through $\alpha_i, \beta_i, \gamma_i$ form a triangle $A_iB_iC_i$ whose sides are parallel to those of ABC turned through the angle θ. Hence the triangles $\beta_i\gamma_iA_i$, $\beta_j\gamma_jA_j$ have parallel sides and it follows from the above that A_i, A_j, A_0

are collinear, and similarly for B_i, B_j, B_0 and C_i, C_j, C_0. From this follows that the triangles $A_iB_iC_i$, $A_jB_jC_j$ have a homothetic center $\phi(\theta)$ which is fixed for all directions of the parallels through A, B, C and the given θ. Hence there exists one direction for the parallels $A\alpha$, $B\beta$, $C\gamma$ for which the straight lines $\alpha_iB_iC_i$, $\beta_iC_iA_i$, $\gamma_iA_iB_i$ meet in the point $\phi(\theta)$. We show next that $A\alpha$, $B\beta$, $C\gamma$ in this case make the angle $-\theta$ with Δ.

Let the x-axis of rectangular coordinates be along Δ and let the abscissas of α, β, γ be denoted by these letters, where here $A\alpha$, $B\beta$, $C\gamma$ make the angle $-\theta$ with Δ, $m=\tan\theta$. It will suffice to prove that αB_1C_1, βC_1A_1, γA_1B_1 which make the angle θ with BC, CA, AB are concurrent in a point, which must be $\phi(\theta)$. Let the coordinates of A be x_1, y_1 and the slope of BC be m_1, and similarly for the other two vertices and sides. We then have $m\alpha=y_1+mx_1$, and

$$(1)\quad \alpha B_1C_1:\ m(1-mm_1)y-m(m+m_1)x+(m+m_1)(y_1+mx_1)=0,$$
$$m[x_3-x_2-m(y_3-y_2)]y-m[m(x_3-x_2)+y_3-y_2]x$$
$$+m^2x_1(x_3-x_2)+m[y_1(x_3-x_2)+x_1(y_3-y_2)]+y_1(y_3-y_2)=0,$$

where the two remaining equations are obtained by cyclic permutations of subscripts and of the letters in the designations of the equations. It is easily verified that the sum of the left members of the three equations is identically zero, and this proves that the three straight lines meet in a point which must be $\phi(\theta)$.

The point $\phi(\pi/2)$ is the known orthopole of Δ with respect to ABC, and $\phi(0)$ is the point at infinity in the directions of the Newton line (ABC, Δ), see the solution of 3890 [1943, 267]. If $m_1=\tan\psi_1$, then $\phi(-\psi_1)=A_0$; for, let the parallels $A\alpha$, $B\beta$, $C\gamma$ make the angle ψ_1 with Δ, then $\beta=\gamma=A_0$ and αB_1C_1, which makes the angle $-\psi_1$ with BC, must coincide with Δ. Thus Δ cuts the locus of $\phi(\theta)$ in A_0, B_0, C_0, and if we use m as a parameter for x, y of this locus, y must have the factors $m+m_1$, $m+m_2$, $m+m_3$. The relation for y turns out to be

$$(2)\quad 2m(1+m^2)Sy=-(m+m_1)(m+m_2)(m+m_3)(x_1-x_2)(x_2-x_3)(x_3-x_1),$$

where S is the area of ABC. The equation of the locus has the form

$$(3)\qquad L_1C-L_2^2=0,$$
$$L_1\equiv(m_2y_2-m_1y_1)y+m_1m_2(y_1-y_2)x+(m_2-m_1)y_1y_2$$
$$+m_1m_2(x_1y_2-x_2y_1)$$
$$L_2\equiv m_1m_2(y_2-y_1)y+(m_2y_2-m_1y_1)x+m_1y_1x_2-m_2y_2x_1,$$
$$C\equiv(m_2-m_1)(x^2+y^2)+[y_1-y_2+m_1x_2-m_2x_1+m_1x_1-m_2x_2]x$$
$$+[x_1-x_2+m_2y_1-m_1y_2+m_1m_2(x_1-x_2)]y$$
$$+(m_2-m_1)x_1x_2+x_1y_2-x_2y_1.$$

The straight lines $L_1=0$, $L_2=0$ are perpendicular, and the cubic is tangent to the circle $C=0$ and to $L_1=0$ at their intersections with $L_2=0$.

Maximum Area of a Variable Triangle

4084 [1943, 330]. *Proposed by Otto Dunkel, Washington University*

On the sides A_jA_k of a given triangle $A=A_1A_2A_3$ as bases, directly similar triangles $B_iA_jA_k$ are constructed interiorly giving the triangle $B=B_1B_2B_3$. Show that, if B has the maximum area when the sense of rotation of its vertices is opposite to that for A, the triangles $B_iA_jA_k$ must be isosceles with $\cot\alpha \cot V=3$, where α is the base angle and V is the Brocard angle for A. Determine the form of the triangle B giving the maximum.

Solution by the Proposer. Let $\mathbf{t}_i$ and $\mathbf{u}_i$ be the vectors of A_i and B_i with the origin at the centroid G of the A triangle whose vertices we assume to have the positive sense of rotation $\alpha=\angle A_jA_iB_k$, $0<\alpha<\pi$, and $A_iB_k/A_iA_j=p$, then

$$\mathbf{u}_1 = \mathbf{t}_2 + p\cos\alpha(\mathbf{t}_3 - \mathbf{t}_2) + p\sin\alpha\mathbf{k}\times(\mathbf{t}_3 - \mathbf{t}_2), \tag{1}$$

where $\mathbf{k}$ is a unit vector such that $\mathbf{t}_i$, $\mathbf{t}_j$, $\mathbf{k}$ has the positive sense of rotation, and two other equations are obtained from (1) by cyclic permutations of the subscripts. From these three equations it follows at once that G is also the centroid of the B triangle. Let S_a and S_b denote the area of the A and B triangles; then a long vector computation gives

$$\begin{aligned} S_b/S_a &= 3p^2 - 3p\cos\alpha + 1 - p\sin\alpha\cot V \\ &= 3(p\cos\alpha - \tfrac{1}{2})^2 + 3\left(p\sin\alpha - \frac{\cot V}{6}\right)^2 - \frac{1}{12}(\cot^2 V - 3). \end{aligned} \tag{2}$$

Since $\cot^2 V\geqq 3$, the minimum of the expression on the right is given by $2p\cos\alpha=1$, $6p\sin\alpha=\cot V$. Hence for the minimum the triangles $B_kA_iA_j$ are isosceles with the base angle α such that $\cot\alpha\cot V=3$.

Let b_i denote the length of side B_jB_k; then another rather long computation gives

$$\left(\frac{b_i}{GA_i}\right)^2 = \frac{\cot^2 V - 3}{4}. \tag{3}$$

Thus the minimizing B triangle is inversely similar to the triangle GA_1A_3', where A_3G is extended to A_3' so that $GA_3'=A_3G$. The area S_b in (2) is negative because of the sense of rotation of the vertices; hence a minimum rather than a maximum is desired.

Equilateral Hyperbola and Circle

4085 [1943, 330]. *Proposed by V. Thébault, San Sebastián, Spain*

Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

Solution by Howard Eves, Syracuse University. Take O as origin of rectangular coordinates, and let the coordinate axes be parallel to the asymptotes of (H). Suppose ω has coordinates (k, h). Then the equations of (O) and (H) may be respectively taken as

$$(1) \qquad x^2 + y^2 - (h^2 + k^2) = 0, \qquad xy - kx - hy + c = 0.$$

The necessary and sufficient condition that an infinity of triangles circumscribe a conic S and be inscribed in a conic S' is, in terms of the customary invariant symbols (see art. 376 of Salmon's *A Treatise on Conic Sections*),

$$(2) \qquad \Theta^2 - 4\Delta\Theta' = 0.$$

Taking S and S' to be (O) and (H) above we find, by substituting in (1),

$$(3) \qquad c^2 + 4(h^2 + k^2)(-h^2/4 - k^2/4 + h^2/4 + k^2/4) = 0,$$

whence we must have $c=0$, or (H) must pass through O.

Let R, S, T, U be the points where (O) cuts the coordinate axes, and let V, W be the points where (O) cuts (H). Then the envelope of the sides of the triangles inscribed in (H) and circumscribed to (O) is the circle (O) minus the six points R, S, T, U, V, W. For these points the triangle degenerates.

Editorial Note. Another proof will be given. If there exists one triangle inscribed in (H) which circumscribes (O), then it is known that we can obtain another such triangle ABC by taking the vertex A as any point on (H) not also on (O) and the two tangents from A to (O) meeting (H) again in B and C; then BC must also be tangent to (O). Now consider the case of A at a point at infinity on (H), say in the direction of the asymptote parallel to the x axis; then in the notation of the above solution the equation of BC is

$$(hk - c)(y + h) - k^2(x - k) = 0.$$

In order for this line to be tangent to (O) the length of the perpendicular from O to it must be $(h^2+k^2)^{1/2}$, and this gives $k^2c^2=0$. It is shown below that k is not zero and hence $c=0$. Conversely, if O is on (H), the above equations show that there is a degenerate triangle of the above kind, and therefore an infinite number of the desired triangles. It will be seen that the theorem is true if we interchange the roles of the centers ω and O.

It is simpler to show synthetically that k is not zero; and in doing this it is interesting to make the whole proof synthetic. With the above notation for the degenerate triangle ABC, let T and T' be the points of contact with (O) of the parallels AB and AC. At least one of the two, say AC, meets (H) in a finite point C, and CB touches (O) in T''. Hence B must also be a finite point; and in the above proof, if k were zero, we must have $T \equiv \omega$, and this leads to the contradiction that $B \equiv A$. Thus (O) is the inscribed or an escribed circle of ABC, and therefore it is tangent to the nine point circle of ABC which is here the straight line through A', the midpoint of BC, and $\overline{C}$, $\overline{B}$, the feet of the parallel altitudes on AB, AC. Hence $A'\overline{C}$ is tangent to (O) at ω and ω is the symmetric

of T'' with respect to OA'. It is easily seen that $A'B\overline{C}$ is isosceles; and, if $T\omega$ and the asymptotes meet OB in W, U, V, that $WB\overline{C}$ and $WU\omega$ are also isosceles. Hence W is the common midpoint of OB and VU, and O must lie on (H).

Conversely, if O is on (H) and ω on (O), let AT and AT' be tangent to (O) at T and T' where as before A is at infinity on (H), let AT meet (H) again in B, and let the tangent from B to (O) touch it at T'' and meet the line AT' in C. We are to prove that C is on (H), and it will then follow that there are infinitely many of the desired triangles ABC where A may be any point on (H). If the asymptotes meet OB in U and V, where U is on ωA, we know that OB and VU have a common midpoint W, and that WT passes through ω. Let A' be the midpoint of BC and $\overline{C}$, $\overline{B}$, $\overline{W}$ be the symmetrics of C, B, W with respect to OA'. Then from symmetry with respect to OA' and the concurrence of $A'C$, $\overline{W}T'$, (O) in T'', the lines $A'\overline{C}$, WT must be concurrent on (O) in the symmetric of T'', and $A'\overline{C}$ must be the tangent at this point of concurrence. Hence this latter point must be ω, the symmetric of T''; and, if ωU meets BC in S, symmetry shows that $\overline{C}\omega = BS = T''C$. Thus C must lie on (H) and the proof is complete.

It can be shown synthetically that the isogonal conjugate of OT'' with respect to this special triangle ABC is (H), and from this follows that OT'' is the tangent to (H) at O. If we denote the tangents to (H) at O, B, C by t_o, t_b, t_c, then t_o and t_b meet on the line $W\omega T$, t_b and t_c meet on $A'\omega\overline{C}$, and t_c on t_o on $T'\omega E$, where E is the midpoint of OC.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor W. D. Cairns represented the Association at the inauguration of Howard F. Lowry as president of the College of Wooster on October 21, 1944.

Professor L. E. Dix represented the Association at the inauguration of Dr. Homer L. Dodge as president of Norwich University on October 9, 1944.

Professor C. A. Murray of West Texas State Teachers College represented the Association at the inauguration of Dr. W. M. Whyburn as president of Texas Technological College on September 30, 1944.

Assistant Professor G. M. Ewing of the University of Missouri, on leave at the Naval Ordnance Laboratory in Washington, D. C., has been promoted to an associate professorship.

Professor Orrin Frink, Jr., of Pennsylvania State College is on leave in the research division of the Aircraft Radio Laboratory at Wright Field, Dayton, Ohio.

Assistant Professor L. P. Hutchison of The Citadel has been appointed to an instructorship at Pennsylvania State College.

D. F. Johnson of Mercersburg Academy has accepted a position at Belmont Hill School, Massachusetts.

Ernest Johnston of the University of Minnesota has been appointed Associate Mechanical Engineer at the Naval Ordnance Laboratory in Washington, D. C.

P. S. Jones of the University of Michigan has been appointed to an instructorship in the University School of Ohio State University.

Professor S. H. Kimball of the University of Maine has been appointed head of the department of mathematics and astronomy.

H. W. Linscheid of the University of Nebraska has been appointed to an assistant professorship at Eastern New Mexico College.

Dr. A. T. Lonseth of Iowa State College has been appointed to an assistant professorship at Northwestern University.

Visiting Professor Szolem Mandelbrojt of The Rice Institute has been granted leave of absence to serve on a mission in London under General Koenig.

Professor C. A. Messick of Park College has been appointed acting chairman of the department during the absence of Professor H. E. Crull, who is in the United States Navy.

Professor I. F. Neff, head of the department of mathematics at Drake Unisity, has retired with the title of professor emeritus.

Professor J. B. Rosenbach of the Carnegie Institute of Technology has been named secretary of the division of humanistic and social studies.

Assistant Professor C. T. Ruddick of Alliance College has been appointed acting treasurer and acting associate professor of philosophy at Lake Erie College.

Professor Mary E. Sinclair, who has retired as head of the department at Oberlin College, has been appointed visiting professor at Berea College.

Assistant Professor B. M. Stewart of Denison University has been appointed to an assistant professorship at Michigan State College.

Dr. G. B. Thomas, Jr., of Massachusetts Institute of Technology has been promoted to an assistant professorship.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE MATHEMATICAL TABLES PROJECT

The Mathematical Tables Project was set up by the Work Projects Administration in January, 1938, under the sponsorship of the National Bureau of Standards. In March, 1943, the sponsoring agency took over its operation with the support of the Office of Scientific Research and Development, the determination of the program being vested in the Applied Mathematics Panel of the National Defense Research Committee. Under the new auspices, the major emphasis is being placed on computations directly related to the war effort, but it is also planned to devote any available time to the important task of finishing various tables which were under way when the Work Projects Administration was discontinued.

Since its founding, the Project has done an amazing amount of work, employing at times a huge corps of computers and computing machines, working in two shifts. Professor A. N. Lowan has been the director of the Program. An effort has been made to select for tabulation functions of fundamental importance in pure and applied mathematics. In the selection and choice of the most useful range and interval of the argument, there has been a continual exchange of views with outstanding mathematicians, physicists, and engineers, both here and abroad.

Twenty large volumes of tables computed by the Mathematical Tables Project have already been published. Also, five shorter tables have appeared in various scientific journals; these are now available in pamphlet form. Two more volumes, namely, a table of associated Legendre functions and a table of arc sin x, are now in the process of being reproduced by the Columbia University Press. Moreover, in addition to present computations related to problems of war research, some time is being devoted to bringing to completion several other large volumes of spherical Bessel functions, Bessel functions of fractional order, Bessel functions of the second kind for complex arguments, exponential integrals for complex arguments, and Chebyschev polynomials. A considerable amount of work has also been done on Jacobi elliptic functions for real and imaginary arguments.

Tables now available are listed below. All except the last four titles may be obtained from the National Bureau of Standards; the last four volumes have been published by the Columbia University Press.

1. Tables of the exponential function e^x.
2. Tables of circular and hyperbolic sines and cosines for radian arguments.
3. Tables of sines and cosines for radian arguments.

4. Tables of sine, cosine, and exponential integrals. (2 volumes)
5. Tables of natural logarithms. (4 volumes)
6. Tables of probability functions. (2 volumes)
7. Tables of the moments of inertia and section moduli of ordinary angles, channels, and bulb angles with certain plate combinations.
8. Table of sine and cosine integrals for arguments from 10 to 100.
9. The hypergeometric and Legendre functions with applications to integral equations of potential theory.
10. Table of arc tan x.
11. Miscellaneous physical tables: Planck's radiation functions; and electronic functions.
12. Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula.
13. Table of the confluent hypergeometric function and its first derivative.
14. Table of integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$.
15. Table of $Ji_0(x) = \int_x^\infty [J_0(t)/t]dt$ and related functions.
16. Table of coefficients in numerical integration formulae.
17. Table of reciprocals of the integers from 100,000 through 200,009.
18. Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments.
19. Table of circular and hyperbolic tangents and cotangents for radian arguments.
20. Tables of Lagrangian interpolation coefficients.

NEW ARMED FORCES INSTITUTE CATALOGUE

The Army and the Navy have recently released a joint catalogue of the U. S. Armed Forces Institute. The new catalogue describes ways by which self-teaching texts may be effectively used by those in service, and outlines steps which the individual should take to procure an evaluation of his educational experience in terms of school and college credit. The Institute, with its nine overseas branches, is, as stated on the cover of the catalogue, "The Army-Navy School with the World Campus." The magnitude of the endeavor and the nature of the work being done deserve the attention of educators generally. The number of persons taking correspondence courses through the USAFI is already in excess of a quarter of a million, of whom over 50,000 are enrolled in university extension correspondence courses approved by the Institute. An average of more than 2000 persons in the Armed Forces are enrolling daily.

The Institute program consists of four major divisions, namely, Institute correspondence courses, university extension correspondence courses, self-teaching courses, and off-duty classes. The correspondence courses are of the traditional type. The self-teaching courses are especially for those students who prefer to study a subject entirely on their own, or who cannot be reached promptly and regularly by mail. In such a case, the Institute furnishes a textbook "which makes it possible to learn a subject without a teacher." Off-duty classes "are taught with specially prepared textbooks that include tests, drills,

and questions and answers, which follow the same method a teacher would use in a formal class. This makes it possible for a group of men to study a subject without a skilled instructor. (The men) appoint a leader or instructor, and get to work."

One fee, $2.00, enrolls any Army enlisted man or woman for Institute self-teaching and correspondence courses. The same fee applies to all personnel of the Navy, Marine Corps, and Coast Guard. So long as the student does acceptable work, he may continue to enroll for additional Institute self-teaching and correspondence courses without further charge. A commissioned or warrant officer of the Army may enroll for any Institute correspondence course by paying the cost of the course. The fee may be ascertained by writing to the United States Armed Forces Institute, Madison 3, Wisconsin, or to any oversea branch, and will cover the costs of texts, lesson service, and end-of-course test. The Government shares the cost of a university extension correspondence course with enlisted personnel of the Army and with all personnel of the Navy, Marine Corps, and Coast Guard; the Government will pay half the text and tuition fee for each course taken up to the amount of $20.00.

The accreditation service of the USAFI is explained by the new catalogue to service men in the following paragraph.

"The Armed Forces Institute does not itself grant high-school or college credit for courses offered to military personnel. The granting of school or college credit is the responsibility of the educational institution in which you may want to use such credit toward a diploma or a degree. The Institute will, however, help you to get school or college credit for any educational work you may do while you are in the service by submitting to the educational institution you designate, a detailed record of your educational achievement. All the major associations of high schools and colleges in the United States, and hundreds of individual institutions, have voted that special arrangements should be made for returning service personnel who want academic credit for their experience in the service. The special arrangements which have been recommended consist of a plan by which men and women who have been in the service will not be required to prove that they have attended specified courses for so many hours (the usual basis for granting credit), but will be given credit for what they have learned, no matter when or how they learned it. To make these arrangements effective, the Armed Forces Institute will give you special tests designed to measure what you have learned in the service. In addition, the Institute will prepare for you a record of the Institute and college or university extension courses you have studied, the service schools you have attended, and the service jobs you have had. On your request, it will forward a report on your examinations and a copy of your educational record to the school or college of your choice, and will obtain for you a statement of the amount of academic credit which will be granted you. If, therefore, you wish to obtain school or college credit for your work, write to the Institute headquarters or to the nearest Institute branch for an application for accreditation service."

Twenty-seven courses in mathematics are now offered by the USAFI. Only eleven of these courses, however, are regarded by the Institute as being upon the college level; they are plane trigonometry, college algebra, plane analytic geometry, descriptive geometry, spherical trigonometry, differential calculus, integral calculus, solid analytic geometry, differential equations, and engineering mathematics (parts I and II). All of the self-teaching texts in mathematics which have been developed up to the present time are in elementary or secondary mathematics.

DISPOSAL OF SURPLUS WAR PROPERTY

Many persons recall opportunities after the last war to secure surplus or used war material at a very reasonable cost. In view of the highly technical nature of the present conflict, it has seemed reasonable to assume that educational institutions might find it possible to obtain many desirable items of equipment at the conclusion of the war.

Important legislation on the disposal of surplus war property has been before Congress for some weeks. On August 22, the House passed the Colmer bill (H.R. 5125) which would set up a Surplus War Property Administration under the direction of one person, the administrator. He would determine "such policies governing prices and other terms and conditions of disposal . . . as he deems necessary to effectuate the objectives and policies of this Act." Moreover, it is provided that non-salable material may be donated to "any agency or institution supported by the Federal Government or any state or local government, or to any non-profit educational or charitable organization . . . ." Among the purposes of the act is the specification "To afford public, governmental, educational, . . . institutions . . . an opportunity to fulfill their legitimate needs."

Two bills on surplus war property have appeared in the Senate, one offered by Senators Stewart, Murray, and Taft (S. 2065), and the other by Senator Johnson (S. 2045). On August 23, a substitute version of S.2065 was reported out. This bill provides for a Surplus Property Board of eight members instead of a single administrator. Parts of Section 12 affecting educational institutions are quoted below.

"The Board may prescribe regulations for the disposition of surplus property to States, and political subdivisions thereof, including municipalities, and to tax-supported and non-profit institutions, as follows:

"(a) Surplus property that is appropriate for school, classroom or other educational use may be transferred to the Federal Security Agency for donation to the States and their political subdivisions and tax-supported educational institutions, and, . . . with the approval of the Board, to other nonprofit educational institutions which have been held exempt from taxation under section 101 (6) of the Internal Revenue Code.

"(b) Surplus medical supplies and equipment may be transferred to the Federal Security Administration for donation to the States and their political subdivisions and to tax-supported medical institutions, and, within rules and

regulations to be prescribed by the Federal Security Administrator, with the approval of the Board, to hospitals or other similar institutions not operated for profit which have been held exempt from taxation under Section 101 (6) of the Internal Revenue Code, and to the American Red Cross.

"(c) Any surplus property may be sold or leased to States, political subdivisions thereof, including municipalities, tax-supported institutions, and nonprofit charitable, medical and educational institutions which have been held exempt from taxation under Section 101 (6) of the Internal Revenue Code, at discounts not to exceed 50 per centum of the sale or lease market value thereof, as the case may be, or 50 per centum of the highest price offered by any private purchaser or lessee, whichever is lower."

It is also specified in Section 12 that non-salable material may be donated to institutions. The Surplus Property Board exercises some control over property disposed of under Section 12 for a period of two years, especially "with respect to the maintenance of the property, (and) its continued use for the general purpose for which it was acquired . . . ."

MATHEMATICIANS NEEDED IN FEDERAL SERVICE

The Civil Service Commission has announced an examination for mathematicians to fill positions in various places throughout the country. At present there are needs at Langley Field in Virginia, in Aberdeen, Maryland, a few vacancies in Cleveland, Ohio, and some in Washington, D. C. Mathematicians are needed to apply their technical training to the solution of problems in ballistics, aerodynamics, vibration, and other scientific problems. The salaries range from $2,433 to $4,428 a year, including overtime pay.

In general, applicants for the $2,433 positions must have had at least 3 years of technical experience in mathematics requiring the applications of the principles of algebra, trigonometry, analytical geometry, and calculus. For the higher-grade positions, applicants must show additional experience of a progressively higher level.

College study which included courses in higher mathematics may be substituted for the prescribed experience on the basis of 1 year of academic study for each 9 months of experience. Graduate study in mathematics may be substituted year for year for the experience prescribed up to a maximum of three years.

No written test will be given and there are no age limits. Appointments to Federal positions are made in accordance with War Manpower Commission policies and employment stabilization programs.

Full information and application forms may be secured at first- and second-class post offices, from the Commission's regional offices, or direct from the U. S. Civil Service Commission, Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SEVENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-seventh summer meeting of the Mathematical Association of America was held at Wellesley College, Wellesley, Massachusetts, on Saturday, August 12, 1944, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Two hundred and seventy-two persons were in attendance at the meetings, including the following one hundred and forty-one members of the Association.

C. R. ADAMS, Brown University
LOUISE ADAMS, High Point College
R. P. AGNEW, Cornell University
H. E. ARNOLD, Wesleyan University
K. J. ARNOLD, Columbia University Statistical Research Group
L. A. AROIAN, Hunter College

D. H. BALLOU, Middlebury College
HELEN P. BEARD, Newcomb College
R. F. BELDING, Vermont Academy
GARRETT BIRKHOFF, Harvard University
G. D. BIRKHOFF, Harvard University
R. P. BOAS, JR., Harvard University
J. G. BOWKER, Middlebury College
H. W. BRINKMANN, Swarthmore College
E. T. BROWNE, University of North Carolina
R. E. BRUCE, Boston University
R. C. BUCK, Harvard University
SISTER LEONARDA BURKE, Regis College, Massachusetts

W. D. CAIRNS, Oberlin College
R. H. CAMERON, Massachusetts Institute of Technology
B. H. CAMP, Wesleyan University
MILDRED E. CARLEN, Brown University
W. B. CARVER, Cornell University
W. F. CHENEY, JR., University of Connecticut
MARY D. CLEMENT, Wells College
NANCY COLE, Connecticut College
J. B. COLEMAN, New York, N. Y.
T. F. COPE, Queens College
LENNIE P. COPELAND, Wellesley College
RICHARD COURANT, New York University
J. H. CURTISS, Bureau of Ships, Navy Department

MARGUERITE D. DARKOW, Hunter College
F. F. DECKER, Syracuse University
C. E. DIMICK, U. S. Coast Guard Academy
ARNOLD DRESDEN, Swarthmore College
WILLIAM H. DURFEE, Yale University
P. S. DWYER, University of Michigan

J. E. EATON, Queens College
H. S. EVERETT, University of Chicago

W. H. FAGERSTROM, College of the City of New York
WILL FELLER, Brown University
W. C. FOREMAN, U.S.N.R.
R. M. FOSTER, Polytechnic Institute of Brooklyn
J. S. FRAME, Michigan State College
PHILIP FRANKLIN, Massachusetts Institute of Technology

C. A. GARABEDIAN, Wheaton College
R. E. GASKELL, Brown University
H. M. GEHMAN, University of Buffalo
B. H. GERE, U. S. Naval Academy
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
W. H. GOTTSCHALK, University of Pennsylvania
V. G. GROVE, Michigan State College

P. R. HALMOS, Syracuse University
G. E. HAY, Brown University
OLIVE C. HAZLETT, University of Illinois
G. A. HEDLUND, University of Virginia
M. R. HESTENES, University of Chicago
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College

M. GWENETH HUMPHREYS, Newcomb College
C. C. HURD, U. S. Coast Guard Academy

R. N. JOHANSON, Boston University
B. W. JONES, Cornell University

G. K. KALISCH, Cornell University
WILFRED KAPLAN, Brown University
J. L. KELLEY, Aberdeen Proving Ground
L. M. KELLY, U. S. Coast Guard Academy
S. H. KIMBALL, University of Maine
J. R. KLINE, University of Pennsylvania
MAURICE KRAITCHIK, New School for Social Research

MARY E. LADUE, Barnard College
A. L. LANCKTON, U. S. Coast Guard Academy
J. A. LARRIVEE, U. S. Naval Observatory
JOSEPH LEHNER, Kellex Corporation

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
INGO MADDAUS, JR., Massachusetts Institute of Technology
JOHN MANDEL, Brooklyn, New York
R. J. MARCOU, Boston College
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
V. O. MCBRIEN, College of the Holy Cross
N. H. MCCOY, Smith College
EDITH A. MCDOUGLE, University of Delaware
REV. P. H. MCGRATH, St. Peter's College
E. J. MCSHANE, Aberdeen Proving Ground
A. E. MEDER, JR., New Jersey College for Women
E. B. MODE, Boston University
C. N. MOORE, University of Cincinnati
R. K. MORLEY, Worcester Polytechnic Institute

C. A. NELSON, New Jersey College for Women
P. B. NORMAN, Columbia University

C. O. OAKLEY, Haverford College
RUFUS OLDENBURGER, Illinois Institute of Technology

GORDON PALL, McGill University
A. E. PITCHER, Aberdeen Proving Ground
A. L. PUTNAM, Yale University

J. F. RANDOLPH, Oberlin College
W. R. RANSOM, Tufts College
C. H. RAWLINS, JR., U. S. Naval Academy
MINA S. REES, Hunter College
C. F. REHBERG, New York University
C. E. RHODES, Union College
HARRIS RICE, Worcester Polytechnic Institute
R. G. D. RICHARDSON, Brown University
H. A. ROBINSON, U. S. Military Academy
SELBY ROBINSON, College of the City of New York
R. E. ROOT, U. S. Naval Academy
P. C. ROSENBLOOM, Brown University
LULU L. RUNGE, University of Nebraska
HELEN G. RUSSELL, Wellesley College

RAPHAEL SALEM, Massachusetts Institute of Technology
HENRY SCHEFFÉ, Syracuse University
W. A. SHEWHART, Bell Telephone Laboratories
M. F. SMILEY, U. S. Naval Academy
C. V. L. SMITH, U. S. Naval Academy
ANDREW SOBCZYK, Massachusetts Institute of Technology
E. R. STABLER, Hofstra College
MARION E. STARK, Wellesley College
E. P. STARKE, Rutgers University
H. W. STEINHAUS, Equitable Life Assurance Society
RUTH W. STOKES, Winthrop College
M. H. STONE, Harvard University
J. S. STUBBE, University of Cincinnati
OTTO SZASZ, University of Cincinnati

J. D. TAMARKIN, Brown University
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University

H. S. VANDIVER, University of Texas
ANDREW VAZSONYI, Harvard University

G. L. WALKER, Cornell University
R. M. WALTER, New Jersey College for Women
A. H. WHEELER, Clark University
J. H. WHITE, Summit, New Jersey
D. V. WIDDER, Harvard University
H. A. WOOD, Chance Vought Aircraft

MABEL M. YOUNG, Wellesley College

S. D. ZELDIN, Massachusetts Institute of Technology

Dormitory rooms were available for members of the organizations and their families in Tower Court, and excellent meals were served in the dining room of this same building. A reception for the visitors was given by the Administration of Wellesley College on Saturday evening in the Recreation Building; and on Sunday evening Professor Barnett of the Wellesley Music Faculty gave a piano recital in Billings Hall.

The joint dinner for the three organizations was held on Sunday evening at 6:30. Professor D. V. Widder of Harvard University acted as toastmaster, and he introduced Captain Mildred H. McAfee, President of Wellesley College and head of the WAVE organization, who gave a most happy address of welcome. President M. H. Stone of the Mathematical Society was introduced and spoke of the contribution which American mathematicians had made toward the prosecution of the war. Professor J. S. Frame then presented resolutions expressing the thanks and appreciation of the visiting organizations to President McAfee and the administration of Wellesley College, to Miss Copeland, chairman of the Department of Mathematics, to Miss Stark, chairman of the Committee on Arrangements, and to the other members of her committee, to Professor Barnett of the Music Department, and to all others who had given of their time and effort to make the meetings so thoroughly enjoyable. The resolutions were adopted by a rising vote.

The American Mathematical Society held sessions on Sunday and Monday. The twenty-sixth Colloquium consisted of four lectures on "Selected topics in the theory of semi-groups" given by Professor Einar Hille of Yale University. On Sunday afternoon, by invitation, Professor C. C. MacDuffee of the University of Wisconsin gave a lecture "On the composition of algebraic forms of higher degree."

The Institute of Mathematical Statistics held sessions on Sunday morning and afternoon in addition to the joint session with the Association on Saturday evening.

The program committee for the Association consisted of Professors J. I. Tracey, chairman; H. M. Gehman and J. J. Gergen. The sessions were held Saturday afternoon and evening.

First Session of the Association

1. "The stability of dynamical systems" by Dr. H. W. Bode, Director of Mathematical Research, Bell Telephone Laboratories.

2. "Some problems of a young instructor" (by one who has been), Professor Marion E. Stark, Wellesley College.

3. "Some aspects of the theory of functions of several complex variables" by Professor W. T. Martin, Syracuse University.

Joint Session of the Association with the Institute of Mathematical Statistics

Symposium on "Potential opportunities for statisticians and the teaching of statistics." Dr. W. A. Shewhart, President of the Institute, opened the discussion, followed by Professor Harold Hotelling. Professor Milton de Silva Rodriques of Sao Paulo University, Brazil, and Dr. Vaclav Myslivec, Czechoslovak Delegate to the United Nations Interim Commission on Food and Agriculture, were introduced and spoke briefly.

Meetings of the Board of Governors

Eight members of the Board were present at the meetings held Sunday morning and afternoon.

The following twenty-nine persons were elected to membership on applications duly certified:

Rev. H. W. Ball, A.M. (Boston Coll.) Missionary, St. George's Coll., Kingston, Jamaica, B.W.I.

W. E. Beeman, M.S. (N. Tex. State T.C.) Instr., North Texas State Teachers Coll., Denton, Tex.

John Bodnar, M.S. in M.E. (Illinois Inst. of Tech.) Instr., Math. and Physics, Jr. Coll. of Connecticut, Bridgeport, Conn.; Test Engr., Chance Vought Aircraft, Stratford, Conn.

A. V. Boswell, A.M. (Northwestern) Asso. Prof., Tennessee A. and I. State Coll., Nashville, Tenn.

C. H. Brown, Ph.D. (Kansas) Asso. Prof., Central Missouri State Teachers Coll., Warrensburg, Mo.

F. M. Carpenter, A.M. (Illinois) Instr., Missouri School of Mines, Rolla, Mo.

H. S. Christian, Jr. Student, Illinois Inst. of Tech., Chicago, Ill.

R. L. Duncan, A.B. (Tulsa) Instr., Univ. of Tulsa Evening Coll.; Div. Titleman, Carter Oil Co., Tulsa, Okla.

Anselm Fisher, A.M. (New York Univ.) Instr., Macalester Coll., St. Paul, Minn.

Sister Catherine Dorothea Fox, Ph.D. (Boston Coll.) President, Trinity Coll., Washington, D. C.

Aletha C. Gaddis, A.M. (Columbia) Instr., Carleton Coll., Northfield, Minn.

J. H. Giese, Ph.D. (Princeton) Aerodynamicist, Bell Aircraft Corp., Buffalo, N. Y.

F. G. Graff, A.M. (Pittsburgh) Major, CAC. Instr., U. S. Military Acad., West Point, N. Y.

D. B. Houghton, A.M. (Michigan) Instr., Princeton Univ., Princeton, N. J.

C. C. Hurd, Ph.D. (Illinois) Lt., U.S.N.R. U. S. Coast Guard Acad., New London, Conn.

P. A. Lagerstrom, Ph.D. (Princeton) Research mathematician, Bell Aircraft, Niagara Falls, N. Y.

A. L. Lanckton, A.M. (Duke) Lt., U.S.N.R. Instr., U. S. Coast Guard Acad., New London, Conn.

Mrs. Elizabeth C. Lukacs. Instr., Berea Coll., Berea, Ky.

F. L. Miksa. Switchman, Illinois Bell Telephone Co., Aurora, Ill.

E. P. Miles, Jr., Master's (Duke) Lt. (SC), U.S.N.R. Supply Officer.

Dewey Moore. Student, Berea Coll., Berea, Ky.

J. V. Pennington, Ph.D. (Rice) Vice Pres., Reed Roller Bit Co., Houston, Tex.

Valentina Potor, A.M. (Indiana) Instr., Berea Coll., Berea, Ky.

Elizabeth M. Ragland, A.M. (Kentucky) Teacher, Jr. High School, Lexington, Ky.

M. M. RESNIKOFF, M.S. (Chicago) Prof., State Teachers Coll., Minot, N. D.

P. C. ROSENBLOOM, Ph.D. (Stanford) Instr., Brown Univ., Providence, R. I.

J. A. TIERNEY, A.M. (Columbia) Instr., Norwich Univ., Northfield, Vt.

ANDREW VAZSONYI, Ph.D. (Budapest), M.S. (Harvard) Teaching Fellow, Grad. School of Engineering, Harvard Univ., Cambridge, Mass.

SISTER M. CLAUDIA ZELLER, Ph.D. (Michigan) Instr., Coll. of St. Francis, Joliet, Ill.

The Board received and placed on file the Report of the American Mathematical Society and the Mathematical Association of America to the Rockefeller Foundation on the activities of the War Policy Committee.

The Board voted in favor of suitable amendments to the by-laws to abolish institutional memberships in the Association, these amendments to be submitted to the Association at the next annual meeting.

It was voted to make an annual contribution of $350.00 per year for the three years 1945–47 to the support of Mathematical Reviews.

It was voted to contribute $200.00 for the calendar year 1944 to the support of the National Mathematics Magazine.

The question of financial help to Sections in securing outside speakers for Sectional meetings was discussed, and it was voted that such assistance be given under certain circumstances and in restricted amounts. The Secretary was instructed to inform all Section Secretaries with regard to the exact terms of the arrangement.

The President was authorized to appoint a committee to study the question of Regional Governors and their election and to report recommendations to the Board.

The Board passed a resolution giving the Finance Committee power to buy and sell securities for the Association's invested funds.

The Finance Committee reported that an audit of the accounts of the Association for the year 1943 had been made by Mr. Taylert of the Treasurer's office of Cornell University. The audit was accepted and approved by vote of the Board.

The resignation of Professor R. E. Langer as chairman of the Committee on Slaught Memorial Papers was presented and accepted by the Board with an expression of appreciation of the time and work that Professor Langer had given to this project. On nomination of President Cairns, Professor N. H. McCoy was appointed chairman of this committee, and the President was authorized to appoint one or more additional members of the committee.

On nomination of the Executive Committee the Board elected the following Nominating Committee: W. D. Cairns, J. S. Frame, and H. M. Gehman, chairman.

W. B. CARVER, *Secretary-Treasurer*

THE ANNUAL MEETING OF THE KANSAS SECTION

The thirteenth annual meeting of the Kansas Section of the Mathematical Association of America was held on Saturday, April 15, 1944, at Washburn University in Topeka, Kansas. This was a joint meeting with the Kansas Association of Teachers of Mathematics. Professor Paul Eberhart, Chairman of the Section, presided at the morning session, and Mr. Herbert Bishop of Manhattan High School presided at the afternoon session.

The attendance was eighty, including the following twenty-seven members of the Association: R. W. Babcock, Wealthy Babcock, Florence Black, Lucy T. Dougherty, Paul Eberhart, W. H. Garrett, Edison Greer, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, C. T. McCormick, Thirza A. Mossman, Sister Jeanette Obrist, O. J. Peterson, P. S. Pretz, C. B. Read, D. H. Richert, G. W. Smith, E. B. Stouffer, W. T. Stratton, Sister M. Helen Sullivan, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, A. E. White.

At the business meeting the following officers were elected for the next year: Chairman, Edison Greer, Beech Aircraft Corporation; Vice-Chairman, C. T. McCormick, Fort Hays Kansas State College; Secretary, Anna Marm, Bethany College.

The following papers were presented:

1. *Algebraic functions*, by Professor D. H. Richert, Bethel College.

This paper was concerned with the properties of an algebraic function of a single variable. After such a function had been defined, the discussion centered around the relation of the definition to such concepts as the fundamental theorem of algebra, polynomials, algebraic numbers, and algebraic integers. A few examples were presented to illustrate the method by which it may be shown that any function expressible by means of radicals is algebraic.

2. *Mathematics in the navy training program*, by Professor G. W. Smith, University of Kansas.

In discussing the mathematics in the naval training programs at the University of Kansas, Professor Smith stated that each group of machinists' mates or electricians' mates took a twelve week course consisting of arithmetic, some elementary algebra, a little plane geometry, and a little trigonometry. He remarked that the mathematics taught in the V-12 program was substantially that included in the usual engineering curriculum.

3. *Mathematics in pre-radar training*, by Professor A. E. White, Kansas State College.

The pre-radar work at Kansas State College was organized by the department of electrical engineering in the fall of 1943, and continued for about eleven months. The purpose was to train men for radio operation and maintenance, and to prepare them for advanced radio work. Approximately three hundred and fifty men received this training. The work in mathematics consisted of an intensive review of elementary algebra (four hours per day for three days), after

which the trainees were divided into sections and assigned to regular class work in mathematics. The men devoted twelve hours per week for four weeks to algebra and trigonometry, and subsequently six hours per week for two weeks were spent in the study of the calculus.

4. *Mathematics in the army training programs*, by Dean E. B. Stouffer, University of Kansas.

The speaker gave a brief report on the experience of the mathematics department of the University of Kansas with the basic phase of the A.S.T.P. He stated that the care with which the students had been selected made it possible for them to carry a very heavy program of mathematics and science in a highly satisfactory manner.

5. *Mathematics in the manufacture of aircraft*, by Edison Greer, Beech Aircraft Corporation.

Mr. Greer remarked that higher mathematics plays an important part in many phases of the manufacture of aircraft, and that there seem to be few branches of mathematics which do not have a place somewhere in the industry. Numerous details were cited in support of these statements. Analytic geometry and descriptive geometry were said to be particularly useful. It was also remarked that much use is now being made of projective geometry in the construction of curves of the second degree.

6. *Mathematics in the army air force program*, by Professor O. J. Peterson, Kansas State Teachers College, Emporia.

Professor Peterson discussed the mathematical content and the objectives of the army air force college training program. He commented upon the effectiveness of a system of periodic diagnostic testing with follow-up remedial instruction aimed at bringing aviation students to acceptable proficiency in the fundamental mathematical skills. Attention was called to the possible benefits in applying similar technique more generally in civilian courses in colleges and secondary schools.

7. *Placement tests for air corps students*, by Professor C. B. Read, University of Wichita.

The speaker discussed the use of the Kansas Mathematics Test No. IV as a basis of sectioning students in the army air force program. The test, consisting of sixteen questions in simple arithmetic and twenty questions in algebra, was given to entering men. Results were used to divide the men into sections. This method of sectioning was found satisfactory, not only for mathematics classes, but also for other subjects.

8. *The Armed Forces Institute tests*, by Laura Z. Greene, Washburn University, introduced by the Secretary.

This paper dealt with the algebra, trigonometry, and analytic geometry tests published by the United States Armed Forces Institute. It was stated

that the tests have some weaknesses, but that in general they are well planned and will probably be useful either as course examinations or as placement tests for service men who return to college.

9. *Correlation of entrance test scores and term grades*, by Professor W. T. Stratton and Professor J. C. Peterson, Kansas State College.

The authors of this paper have been interested in securing a machine scorable type of examination in mathematics to replace the regular problem type of test, so that the results could be scored quickly and used for guidance and placement of freshmen. Their study showed a correlation of 0.74 between the machine scorable type and the problem type of test. The results of the study indicated that boys are more affected by the war situation than are the girls. The correlation of the first semester grades and the scores on the Iowa Physical Science Aptitude Test was 0.216 for boys and 0.678 for girls.

10. *Random jottings from an instructor's notebook*, by Professor C. B. Read, University of Wichita.

Professor Read described certain items which have been helpful in the presentation of material to students. Topics covered were of wide range, but in most cases had reference to the teaching of elementary mathematics. In particular, emphasis was placed upon the correct usage of terms, the recognition of alternative definitions, and the use of reasonable standards of rigor.

11. *Pre-induction courses in mathematics*, by Edna Austin, Topeka High School, introduced by the Chairman.

The speaker outlined the mathematics needed by men in the army, and the content of manuals prepared for the use of people in the military forces.

12. *The mathematics program in the high school at present and in the post-war period*, by T. J. LaRue, Junction City High School, introduced by the Secretary.

In this address the speaker stressed the need for a well-rounded, flexible mathematics program in the high schools. He said that such a program will result from following the recommendations of the M.A.A. and the National Council committees, and that those schools which have followed these recommendations have had little difficulty in meeting the needs of war time. It was stated that the principles which are stressed now will continue to be important after the war.

13. *Report of the committee on the improvement of instruction*, by Professor Gilbert Ulmer, University of Kansas.

This paper constituted the report of a committee appointed a year ago to study the problems confronting the mathematics teachers of Kansas. Professor Ulmer described certain problems which had been considered by the committee, and presented a recommendation that the Kansas State Department of Education be requested to establish a minimum requirement of one unit of mathe-

matics for graduation from accredited high schools of the state. The recommendation was adopted.

ANNA MARM, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 6, 1944. Sessions were held in the afternoon, at dinner, and in the evening. Professor Tibor Radó, Chairman of the Section, presided at these sessions.

Forty-five persons registered attendance, including the following thirty-four members of the Association: F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, C. T. Bumer, V. B. Caris, Nancy Cole, Rufus Crane, Wayne Dancer, B. C. Glover, Mary A. Goins, R. C. Hildner, Margaret E. Jones, L. C. Knight, A. C. Ladner, Edith J. McKissock, E. S. Manson, C. G. Maple, E. J. Mickle. C. C. Morris, Tibor Radó, J. F. Randolph, S. E. Rasor, R. B. Rice, L. D. Rodabaugh, K. C. Schraut, Mary E. Sinclair, B. M. Stewart, J. L. Synge, E. P. Vance, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, J. B. Brandeberry, University of Toledo; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of the Executive Committee, H. M. Beatty, Ohio State University; Member of the Program Committee, H. M. MacNeille, Kenyon College. It is expected that the next meeting will be held at the Ohio State University on Thursday, April 5, 1945.

The following program was presented:

1. *On triangulation*, by Professor Tibor Radó, Ohio State University.

2. *Two rectangles in a quarter circle*, by Professor B. M. Stewart, Denison University.

In the first-quadrant quarter circle bounded by $x^2+y^2=1$, $x=0$, and $y=0$, consider the rectangle A_1 bounded by $x=0$, $x=\cos a$, $y=0$, $y=\sin a$, and the rectangle A_2 bounded by $x=0$, $x=\cos b$, $y=\sin a$, $y=\sin b$, subject to the restriction that $0<a<b<\pi/2$. The figure is related to a motor design problem arising in electrical engineering, where it is desired to maximize the area $A=A_1+A_2$ under the condition $A_1=A_2$. One feature of the problem is that the function $A=\sin 2a$ has at $a=\pi/4$ a maximum, discoverable by the elementary calculus, but not acceptable as a solution. The other feature is the variety of methods from the theory of equations and from elementary and advanced calculus which can be used to obtain the approximate solution $a=18°2'27''$ and $b=52°4'10''$.

3. *Demonstration of the new navy plotting board*, by Professor C. O. Williamson, College of Wooster.

Four problems were employed to demonstrate the rapidity with which the board enabled one to solve vector triangles in navigation.

4. *The life and work of Sir William Rowan Hamilton*, by Professor J. L. Synge, Ohio State University.

The speaker placed particular emphasis on Hamilton's contributions to the calculus of variations. The central idea which motivated Hamilton's early work (until his invention of quaternions) was the characteristic, or principal, function. By characterizing an optical or dynamical system by means of one such function, Hamilton placed himself in the category of men such as Descartes, Lagrange, and Laplace. For the major contribution of each of these consisted in a similar simple characterization of a general problem. The paper dealt also with Hamilton's life, his remarkable attainments as a child, his appointment to professorial rank while still an undergraduate, the many academic distinctions he received, and his habits of working.

5. *Development of college mathematics in Ohio in the past fifty years*, by Professor Mary E. Sinclair, Oberlin College.

6. *Future development of college mathematics in Ohio*, by Professor I. A. Barnett, University of Cincinnati.

The last two papers formed the subject of a round table discussion at the evening session.

RUFUS CRANE, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, December 9, 1944
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274, and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

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